Analysis of Premiums and Reserves in Mulitple Life Insurance using Copula

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The dependence between the insureds in multiple-life insurance contracts is studied. With the future lifetimes of the insureds modeled as correlated random variables, both premium and reserve are different from those under independence. In this paper, Gaussian copula is used to impose the dependence between the insureds with Gompertz marginals. At various dependence levels we analyze the change of the premiums and reserves of standard multiple-life insurance contracts. We find that, for some contracts, the insurance quantities based on the assumption of dependent lifetimes are quite different from those under independence as its correlation increase, which elucidate the importance of dependence model in multiple-life contingencies in both theory and practice.

KEYWORDS: Gaussian copula, reserves analysis, multiple life insurance, joint life survival function.

Chapter 1

Intoduction

A multiple-life insurance contract involves more than one life and pays benefits depending on multiple life status of the insured lives. In the traditional actuarial literature, the future lifetimes, or times-till-death, of the lives involved in multiple-life contracts have been considered to be independent. More recently, the independence assumption has been relaxed (e.g., Frees et al. (1996)) and there have been attempts to impose dependence structure via multivariate distributions or copulas. Dependence between lives in joint life contracts can arise from, e.g., common accidents or life style. As net premiums of insurance contracts are written as expected values of the function of the future lifetime random variables, the assumption of independece may lead to an inadequate actuarial pricing and reserve.

When it comes to multivariate models, the underlying dependence structure plays a crutial role in the calculation of many distributional quantities including the moments. Since copula was developed by Sklar (1973), it has been applied in many academic fields. Zhang and Singh (2006) analyze a bivariate distribution of flood peak and volume by using a copula method in the civil engineering field. Scholzel and

Friederichs (2008) studied various multivariate random variables in climate research, for example, bivariate distribution of average daily precipitation and minimum temperature or bivariate distribution of daily wind maxima between two subway stations. Onken et al. (2009) addopted copula method for analyzing simultaneous spike-counts in biology. In quantative finance field, Li (2000), one of famous actuaries in the world, introduces a new method of pricing framework for credit risk derivertives, making a sensation among the practicians in the financial markets. Copulas have been also applied in the actuarial science field by many scholars for the long time. Frees et al. (1996) evaluate a multiple life annuity value using Frank copula, and Shemyakin and Youn (2006) analyze joint last survivor insurance contract using a copula model. Shi and Frees (2011) and de Jong (2012) selected copula method for modeling dependency of the loss triangles to analyze general insurance reserves.

To this extent, the present article investigates the impact of the dependence between the mortalities in multiple life contracts. Gaussian copula is used to model the dependency because of its remarkable advantages. As suggested by Wang (1998), only Gaussian copula allows an arbitrary correlation matrix yet still lends itself to efficient simulation techniques. Moreover, the dependent parameter of Gaussian copula, ρ, is easy to understand since it has a same interpretation with the coefficient of correlation in statistics. Therefore, we use Gaussian copula with the marginals modelled by Gompertz mortality. Both pricing and the reserving under this copula model are examined , pricing first, and then reserving .

The article is organized as follows. In chapter 2, we review the random variables used in joint life models and introduce Gaussian copula which is used to calculate the prices and reserves of joint life contracts under dependence. We also present an alternative explanation for the relationship between the multiple-life random variables analysed by Youn et al. (2002), and re-confirm their result using the copula method. In chapter 3, we study some formulae of premiums and reserves under independence. Chapter 4 and 5 show how the copula model alters the traditional premium and reserve values calculated under independence. In particular, we analyze the four standard multiple-life insurance contracts which serve as basic building blocks for contract analyses, which is the topic of Chapter 6. Most of the backgrounds of actuarial science field are based on Bowers et al. (1997) and Dickson et al. (2009), and the backgrounds of copulas on Nelsen (2006), Klugman et al. (2010) and Cherubini et al. (2004).

Chapter 2

The Future Lifetimes R.V.s and Gaussian copula

2.1 Random variables for multiple-life status

Following the notations from Bowers et al. (1997), we denote the continuous random future lifetime of a male aged x by T_x. In particular, we further denote $T_0 = X$, a random variable representing the future lifetime of the male at birth. Then T_x can be written as a conditional random variable

$$T_x = X - x|X > x. \tag{2.1}$$

Similarly, We can define T_y and Y for a female aged y. In the spousal mortality context, we define the multiple life random variables T_{xy} and $T_{\overline{xy}}$ by

$$T_{xy} = min(T_x, T_y), \quad T_{\overline{xy}} = max(T_x, T_y). \tag{2.2}$$

Note that both T_{xy} and $T_{\overline{xy}}$ assume (x) and (y) are still alive at that time. We can also write these random variables as

$$T_{xy} = min(X - x|X > x, Y - y|Y > y) \tag{2.3}$$

and

$$T_{\overline{xy}} = max(X - x|X > x, Y - y|Y > y). \tag{2.4}$$

From the definition above, we have a well-known identity

$$T_{xy} + T_{\overline{xy}} = T_x + T_y \tag{2.5}$$

without mentioning any dependence assumption between (x) and (y). Furthermore, standard textbooks such as Bowers et al. (1997) also gives us

$${}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x +_t p_y \tag{2.6}$$

and

$$\overline{A}_{xy} + \overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y. \tag{2.7}$$

Equation (2.7) is widely used for calculating multiple-life insurance premium. However, Youn et al. (2002) show that, if independence assumption is relaxed, equation (2.5) is not precisely true, and equations (2.6) and (2.7) do not hold. They verify the fact by using Hougaard copula with Weibull marginal. Actually, we can use a slightly different argument from Youn et al. (2002) to prove that equation (2.6) fails to hold in general as follows.

The left side of (2.6) is written as

$$
\begin{aligned}
{}_tp_{xy} + {}_tp_{\overline{xy}} &= P[X > x + t \; and \; Y > y + t | X > x, Y > y] \\
&\quad + P[X > x + t \; or \; Y > y + t | X > x, Y > y] \\
&= P[X > x + t | X > x, Y > y] + P[Y > y + t | X > x, Y > y] \\
&= P[T_x > t | T_y > 0] + P[T_y > t | T_x > 0], \qquad (2.8)
\end{aligned}
$$

where the second equality comes from the probability set operation $P[A \cup B] = P[A] + P[B] - P[A \cap B]$. The right side of (2.6) is simply

$$
\begin{aligned}
{}_tp_x +_t p_y &= P[X > x + t | X > x] + P[Y > y + t | Y > y] \\
&= P[T_x > t] + P[T_y > t]. \qquad (2.9)
\end{aligned}
$$

Hence it is clear that the conditions given inside (2.8) and (2.9) are not identical. When X and Y are independent, (2.8) reduces to (2.9), and both yield the identical value. This discrepancy is necessarily carried over to (2.7) as well. To summarize, the proper evaluation of the left sides of (2.6) and (2.7), or multiple life random variables in general, requires the presence of precondition on the other variable, that is,

$$T_x | T_y > 0 \; and \; T_y | T_x > 0. \qquad (2.10)$$

In other words, the correct expression of (2.5) that holds true in general is

$$T_{xy} + T_{\overline{xy}} = T_x | T_y > 0 + T_y | T_x > 0. \qquad (2.11)$$

2.2 Gaussian copula

Let F be a d-demensional joint distribution function (d.f.) with marginals $F_1, ..., F_d$. The theorem of Sklar (1973) asserts that, when the marginals are continuous, a copula C with

$$F(x_1, ..., x_d) = C[F_1(x_1), ..., F_d(x_d)] \tag{2.12}$$

exists uniquely for every $x_1, ..., x_d \in \mathbb{R}$. The copula itself therefore can be obtained from (2.12) for all $u = (u_1, ..., u_d) \in [0, 1]^d$ as

$$C(u_1, ..., u_d) = F[F_1^{-1}(u_1), ..., F_d^{-1}(u_d)]. \tag{2.13}$$

To this extent, we focus on the Gaussian copula with $d = 2$, commonly called the Bivariate Gaussian copula

$$C(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} dxdy, \tag{2.14}$$

where $(u, v) \in [0, 1]^2$ and Φ is the d.f. of the standard normal distribution. If we set $F_{T_x}(t)$ and $F_{T_y}(t)$ as the cumulative distribution functions of T_x and T_y, then, from (2.12), we may write the joint d.f. of the future lifetimes of two lives as

$$\begin{aligned} &F_{T_x,T_y}(t_1, t_2) \\ &= C\left[F_{T_x}(t_1), F_{T_y}(t_2)\right] \\ &= \int_{-\infty}^{\Phi^{-1}(F_{T_x}(t_1))} \int_{-\infty}^{\Phi^{-1}\left(F_{T_y}(t_2)\right)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} dxdy, \end{aligned} \tag{2.15}$$

Note that the dependence between T_x and T_y has been created via the correlation ρ embedded in the Gaussian copula. When $\rho = 0$, (2.15) reduces to the classical independent case where the joint d.f. becomes simply the product the two marginals d.f.'s

$$F_{T_x,T_y}(t_1, t_2) = F_{T_x}(t_1)F_{T_y}(t_2), \tag{2.16}$$

which is the case for the classical independent multiple life framework.

Gaussian copulas have been used in financial field for its good properties. According to Wang (1998), only Gaussian copula allows an arbitrary correlation matrix yet still lends itself to efficient simulation techniques. Therefore, we adopt the Bivariate Gaussian copula to calculate multiple life insurance quatities where the dependence structure exists between the two lives.

2.3 Marginal distribution of Gaussian copula

To determine the Gaussian copula in the previous section, we should set the two marginal distributions of the future lifetimes of insureds. The two marginals of the future lifetimes random variables, T_x and T_y, can be observed from a life table which is used for calculating actuarial quantities. In this paper, Gompertz distribution functions are used to reflect the empirical distributions of the life table, the parameters of Gompertz d.f.s are estimated from the experience life table.

Gompertz d.f., which is a continuous distribution function, allows more convenient and efficient generating process of random samples. Moreover, when the Gaussian copula has continuous marginals, we can assure that there exists unique Gaussian copula which is verified by the Sklar's theorem.

Figure 2.1 shows empirical survival distributions of male and female from 7^{th}

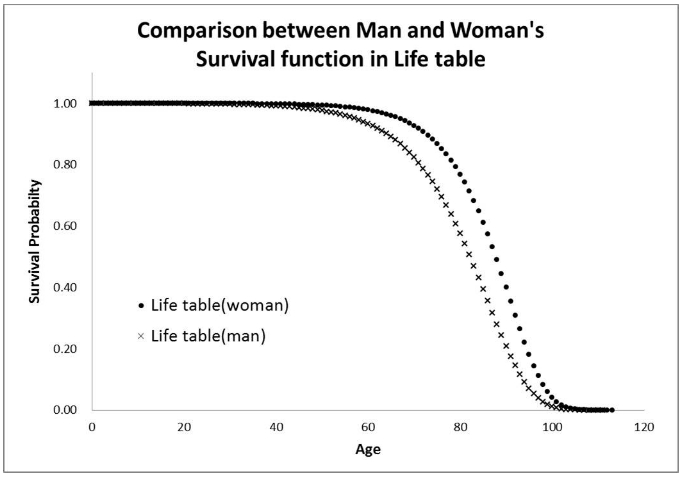


Figure 2.1: Comparison between man and woman's empirical survival functions in the 7^{th} Experience Life Table in Korea

Experience Life Table in Korea. As shown in Figure 2.1, the expected future lifetime of female is longer than that of male. To fit the Gompertz distribution function to the empirical one, the parameters of the Gompertz marginals are estimated by using the least squared method. Gompertz distribution of the future lifetime of person aged x, is given by

$$F_{T_x}(t) = 1 - \exp[e^{(x-m)/\sigma}(1 - e^{t/\sigma})], \tag{2.17}$$

where the mode, m, and the scale measure, σ, are parameters. This formula is suggested by Carriere (1994), which is transformed from the original Gompertz d.f. for straightforward estimation of parameters.

Table 2.1 shows estimates of m and σ of both male and female's distributions. As a measure of comparing the estimated Gompertz d.f. with the empirical survival distributions, we conduct K-S test between the random sample from the estimated distribution and the emprical survival probabilities. The values of test statistic D are provided in Table 2.1. Based on the significance levels 5% and 10%, we cannot

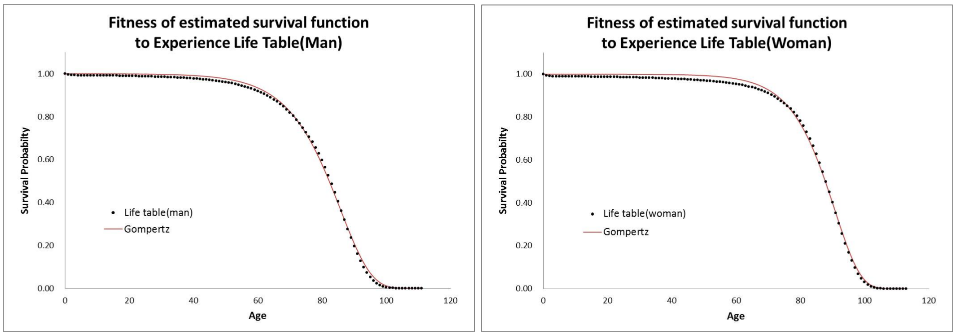


Figure 2.2: Comparison of Experience Life Table and estimated survival function

Table 2.1: Estimated parameters of Gompertz distribution

	Man	Woman
$\hat{m}$	85.69	90.7
$\hat{\sigma}$	9.57	8.01
D	0.09279	0.04608
Critical value	0.122 (α=0.10), 0.136 (α=0.05)	

reject the null hypothesis that the two d.f. are same. Therefore, the empirical survival d.f. can be superseded by the estimated Gompertz d.f.. Figure 2.2 indicates that the estimated distribution adequately explains the empirical survival distribution of Experience Life Table.

Chapter 3

Monte-Carlo simulation with Gaussian copula

As many distributional quantities arising from (2.15) are obtained from Monte Carlo simulation method, we now briefly discuss how random samples are drawn in the presence of the Gaussian copula, both conditionally and unconditioanlly. The algorithm in this chapter plays an important role in the calculations of the multiple life insurance quantities under certain given conditions; the reader is referred to Cherubini et al. (2004) for an in-depth look at the algorithms explained here.

3.1 Simulating dependent lifetimes using Gaussian copula

Before we start to describe how to generate dependent random vectors using Gaussian copula, we present important property of bivariate standard normal distribution. Let

(Z_1, Z_2) be a bivariate standard normal random variable with density function

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2(1-\rho^2)}\right\} \tag{3.1}$$

where $-\infty < z_1, z_2 < \infty$ and $-1 \leq \rho \leq 1$. It is well-known that marginal random variables Z_1 and Z_2 are also standard normal distributed, and so is the conditional distribution of Z_1 given $Z_2 = z_2$:

$$Z_1 | Z_2 = z_2 \ \sim N(\rho z_2, (1-\rho^2)) \tag{3.2}$$

See, e.g., Ross (2006), for the properties of the normal distribution. This conditional distribution result allows a convenient way to simulate samples from the bivariate standard normal distribution in two steps as follows.

1. First, draw two random observations from the independent standard normal, and denote these w_1 and w_2. Set $z_2 = w_2$.

2. In the second step, simulate a sample from the conditional distribution (3.2). This can be done by setting $z_1 = \rho w_2 + w_1\sqrt{(1-\rho^2)}$, where w_2 is from the first step.

The resulting (z_1, z_2) is then a simulated sample from the bivariate standard normal distribuiton with correlation ρ.

We now describe how to generate a random vector (t_1, t_2) from the bivariate Gaussian copula (2.14) with marginal d.f.'s $F_{T_x}(t_1)$ and $F_{T_y}(t_2)$, as stated in (2.15). The procedure requires first to simulate a sample from the copula function (2.12) and then transform the obtained (copula) sample to the corresponding marginals via its inverse d.f. This method is actually quite general and can be applied to any copula

Table 3.1: Generating correlated random vector under Gaussian copula

Correlated standard Normal vector	Convert to uniform random vector	Transform to marginal distribution
$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$	$\begin{pmatrix} \Phi(z_1) \\ \Phi(z_2) \end{pmatrix} \longrightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$	$\begin{pmatrix} F_{T_x}^{-1}(u_1) \\ F_{T_y}^{-1}(u_2) \end{pmatrix} \longrightarrow \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$

and marginals. For our choice of the model, its specific sampling steps are described in Table 3.1. All distributional quantities, such as the expectation value and higher moments, can be calculated from this Monte-Carlo method.

3.2 Simulating conditional lifetime using Gaussian copula

In addition to the unconditional sampling explained above we also need the conditional sampling procedure in the copula model. For example, later we will need to determine $\mathbb{E}[T_x|T_y < t_2]$ for multiple life reserves, which in turn requires an efficient way to sample from conditional random variables, such as $T_x|T_y < t_2$ or $T_y|T_x < t_1$.

We describe the conditional sampling procedure from the Gaussian copula model in several steps where the conditional random variable of interest is $T_x|T_y < t_2$; similar arguments can be made for other conditional random variables.

1. Generate u_1 and u_2 independently from the uniform distribution

2. Calculate $k = F_{T_y}(t_2)$ and obtain $u_2 k$, which is always less than k. This step effectively generates samples from area $T_y < t_2$

3. Let $z_2 = \Phi^{-1}(u_2 k)$, a standard normal sample. Then, from (3.2), we see that

$$Z_1|Z_2 = z_2 \sim N\left(\rho z_2, (1 - \rho^2)\right).$$

Table 3.2: Generating the conditional random sample using Gaussian copula

Determine k generate u_1, u_2	Calculate standard Normal value	Determine value of $Z_1 \vert Z_2 = z_2$	Convert to marginal distribution of $T_x \vert T_y < t_2$
$k = F_{T_y}(t_2)$, u_1, u_2	$z_2 = \Phi^{-1}(ku_2)$ $w_1 = \Phi^{-1}(u_1)$	$z_1 \vert z_2 = w_1\sqrt{(1-\rho^2)} + \rho z_2$	$t_1 = F_{T_x}^{-1}\left[\Phi(z_1 \vert z_2)\right]$

4. Set $w_1 = \Phi^{-1}(u_1)$ using the inverse transform. Then w_1 is a sample from the standard Normal distribution.

5. Get a random sample from the conditional distribution of $Z_1|Z_2 = z_2$ using (3.2):

$$z_1|z_2 = \rho z_2 + w_1\sqrt{(1-\rho^2)}$$

6. Finally, convert $z_1|z_2$ to find the $t_1|t_2$ using

$$t_1|t_2 = F_{T_x}^{-1}\left[\Phi(z_1|z_2)\right]$$

Here $t_1|t_2$ is a sample from the conditional random variable of interest $T_x|T_y < t_2$.

By repeating the steps above one can obtain random samples to determine various values of the conditional distributional quantities. A summarized version of this algorithm is presented in Table 3.2.

3.3 Generating multiple-status random samples

The inverse method is useful for generating ramdom samples from the marginal distributions. If we found a quantile function of the cumulative distribution, random

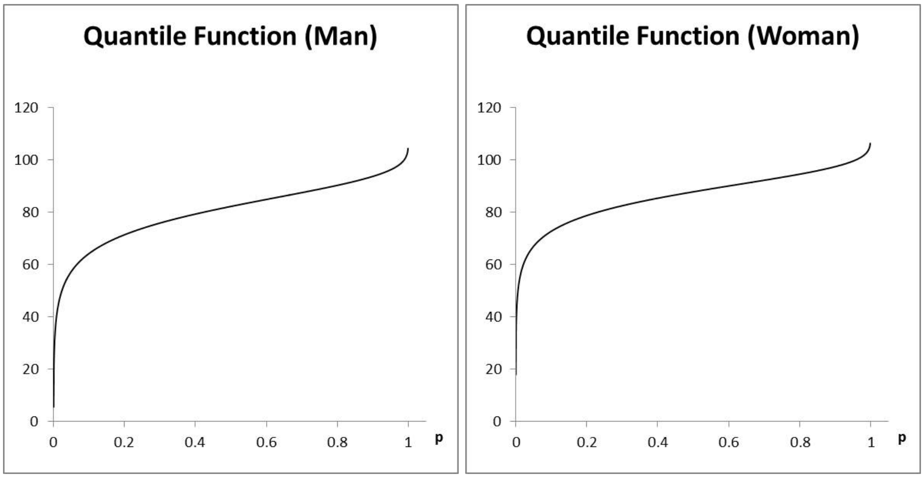


Figure 3.1: Estimated quantile functions from the life table

samples can be generated through the quantile function by using inverse method. The Gompertz d.f., equation (2.17), has its quantile function which can be written as

$$VaR_p\,(T_x) = \sigma \ln \left(1 - \exp\left(-\frac{(x-m)}{\sigma} \right) \ln(1-p) \right), \tag{3.3}$$

where x represents the age of the insured. Figure 3.1 are the estimated quantile functions of man and woman, which the estimates from Table 2.1 are plugged in the equation (3.3).

The future lifetimes of the insureds are correlated random vectors from the joint distribution with its marginals and the copula. To analyze the patterns of random vectors by its dependent structure, the future lifetime of the insureds are generated at various ρ values. It is rational to assume that the future lifetimes of the coupled insured are positively correlated in the multiple insurance context.

Using the simulation method of previous section, we generate the random vectors from the independence dependency level to positively perfect correlation. Figure

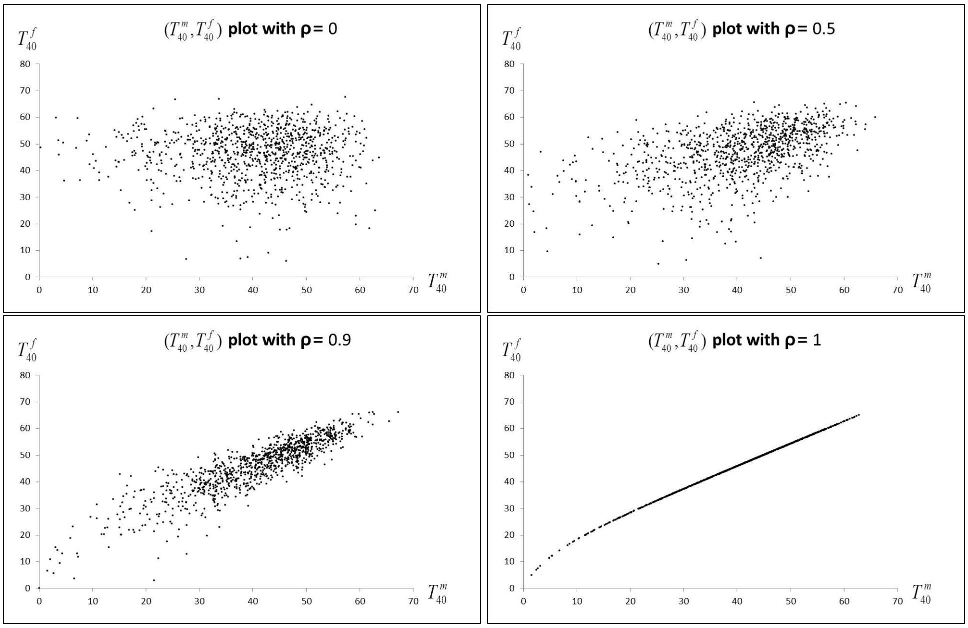


Figure 3.2: Patterns of correlated random vectors (T_x, T_y)

3.2 illustrates the patterns of the future lifetimes of insureds when there exists correlation. In the left upper panel of Figure 3.2, if the future lifetimes of insureds are independent, the future lifetime random vectors are evenly spread out over the graph. In the independent assumption, since the life status of the spouse has no effect on the death of insured, the vectors are randomly generated. As the dependency level goes up, however, the future lifetimes tend to gather in the midle of the graphs because the dependency of the insureds makes the gap between the future lifetime of man and woman small.

In the spousal mortality context, we define multiple life random variables T_{xy} and $T_{\overline{xy}}$ as

$$T_{xy} = min(T_x, T_y), \quad T_{\overline{xy}} = max(T_x, T_y).$$

Table 3.3: Mean and variance of T_{xy} and $T_{\overline{xy}}$

	T_{xy}		$T_{\overline{xy}}$	
$Corr(T_x, T_y)$	Sample Mean	Sample Var.	Sample Mean	Sample Var.
0	37.13131	113.0603	49.99154	49.80457
0.5	38.78795	134.4488	48.35825	73.75241
0.9	40.56953	140.7218	46.67337	104.7848
1	40.78768	131.1679	46.33843	101.4898

T_{xy} represents the future lifetime of the first life to die. In contrast, $T_{\overline{xy}}$ represents the future lifetime of the last life to die. From the definition, T_{xy} and $T_{\overline{xy}}$ have the following relationship.

$$max(T_x, T_y) = min(T_x, T_y) + |T_x - T_y| \quad (3.4)$$

Equation (3.4) implies the maximum of future lifetimes of two insureds equals to the sum of the minimum of future lifetimes of two insureds and the time gap between them. Since the dependence of future lifetime of couple has an effect on the gap, T_{xy} and $T_{\overline{xy}}$ are also affected by the couple's dependency. We generate the samples of T_{xy} and $T_{\overline{xy}}$ at various levels of ρ, given that the ages of male and female are both 40, and calculate their sample means and variances.

In Table 3.3, as the correlation coefficient of T_x and T_y increases, the expected value of T_{xy} also increases, while the expected value of $T_{\overline{xy}}$ decreases. As shown in Figure 3.3, when the future lifetimes of the insureds are independent, they have a low frequency in lower left part of the graph, but as the dependency increases, there are more samples in lower left parts of graphs than the independent case. As the dependency of the insureds increases, the future lifetime of the last life to die tends to decline when the spouse has already died. The result confirms that the expected

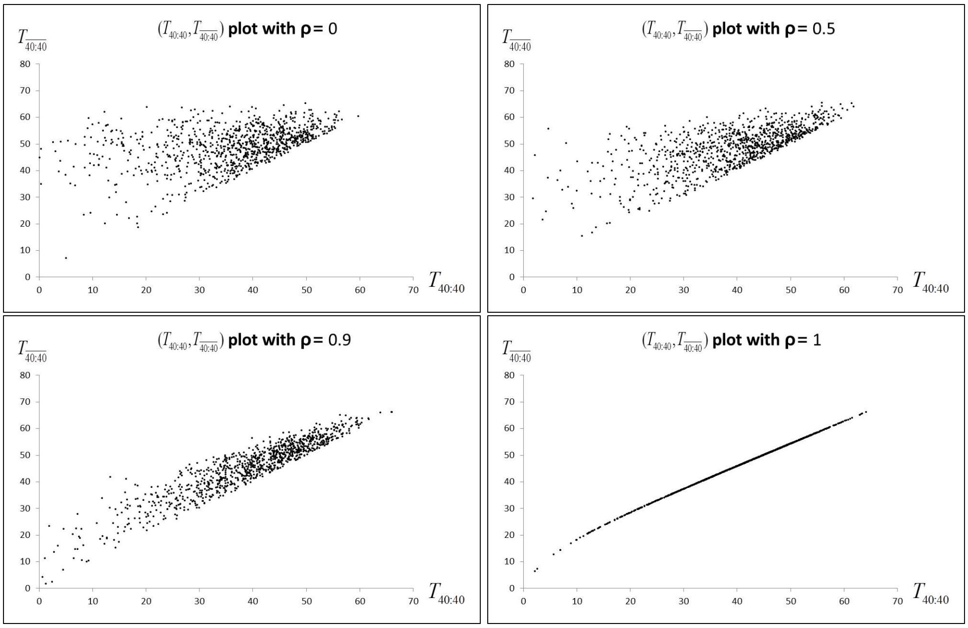


Figure 3.3: Patterns of Correlated random vectors: T_{xy} & $T_{\overline{xy}}$

value of $T_{\overline{xy}}$ decreases as expected. Under the dependent assumption, changes in the expected values, T_{xy} and $T_{\overline{xy}}$, are reconfirmed using cumulative distribution functions(c.d.f) of them. The c.d.f. of T_{xy} can be written as

$$P\left(T_{xy} \leq t\right) = 1 - P\left(\min\left[T_x, T_y\right] > t\right) = 1 - P\left(T_x > t \cap T_y > t\right). \tag{3.5}$$

The last term of Equation (3.5) has the following relationship by the definition of conditional probability.

$$P\left(T_x > t \cap T_y > t\right) = P\left(T_x > t\right) P\left(T_y > t | T_x > t\right) \tag{3.6}$$

In comparison with independent case, the conditional probability $P(T_y > t|T_x > t)$ has larger value when two variables are positivly correlated. Therefore, the values of the c.d.f. of T_{xy} decrease when dependency is assumed and this makes the c.d.f. of T_{xy} relatively more convex. The convexity of c.d.f. implies that the probability of the first death of insureds in short time is shifted right to the larger values, making its mean and variance increase. That is, samples of T_{xy} are more extensively distributed than the independent case. On the other hand, the c.d.f. of $T_{\overline{xy}}$ can be written as

$$P\left(T_{\overline{xy}} \leq t\right) = P\left(\max\left[T_x, T_y\right] \leq t\right) = P\left(T_x \leq t \cap T_y \leq t\right). \tag{3.7}$$

As like Equation (3.6), we can rewrite the last term of Equation (3.7) as

$$P\left(T_x \leq t \cap T_y \leq t\right) = P\left(T_x \leq t\right) P\left(T_y \leq t | T_x \leq t\right). \tag{3.8}$$

The conditional probability that the future lifetime of the woman is less than t given that the husband has already been died, $P(T_y \leq t|T_x \leq t)$, is larger than $P(T_y \leq t)$ which is the case of independent.

3.4 Re-confirmation of the multiple life equalities

Using the actual Korean experienced life tables, we confirm the failure of equation (2.7) using a bivariate Gaussian copula with Gompertz marginals. Estimates from the previous section are plugged in the parameters of Gompertz marginals. For the calculation, age of both male and female are set at 40, and force of interest at 0.12, which are arbitrarily fixed values. Since the expected value of the Monte Calro method always involves errors, we set the numbers of simulation to hundred thousand to keep the simulation errors under 5×10^{-4}. After this section, when the insurance contracts are evaluated using the Monte Calro method, a reference sample size is a hundred thousand.

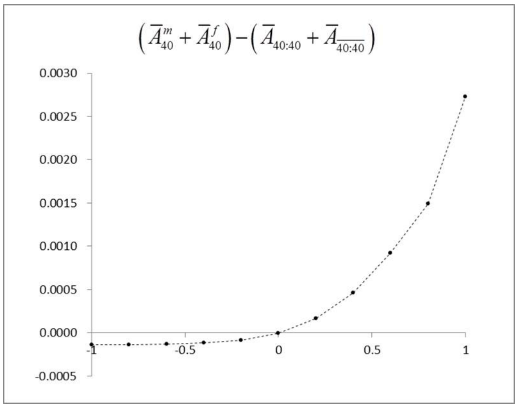


Figure 3.4: Failure of equation (2.7) under dependence. Horizontal axis is correlation

Table 3.4: Failure of equation (2.7) under dependence

					(A)	(B)	
ρ	$\overline{A}_{40}^{m}$	$\overline{A}_{40}^{f}$	$\overline{A}_{40:40}$	$\overline{A}_{\overline{40:40}}$	$\overline{A}_{40}^{m}+\overline{A}_{40}^{f}$	$\overline{A}_{40:40}+\overline{A}_{\overline{40:40}}$	(A)-(B)
-1	0.0259	0.0117	0.0355	0.0023	0.0376	0.0378	-0.0001
-0.8	0.0260	0.0116	0.0352	0.0025	0.0376	0.0377	-0.0001
-0.6	0.0262	0.0115	0.0351	0.0028	0.0377	0.0379	-0.0001
-0.4	0.0260	0.0116	0.0346	0.0031	0.0376	0.0378	-0.0001
-0.2	0.0262	0.0115	0.0343	0.0036	0.0378	0.0379	-0.0001
0	0.0263	0.0116	0.0338	0.0041	0.0379	0.0379	0
0.2	0.0262	0.0116	0.0328	0.0049	0.0379	0.0377	0.0002
0.4	0.0260	0.0115	0.0314	0.0057	0.0376	0.0371	0.0005
0.6	0.0259	0.0116	0.0298	0.0068	0.0375	0.0366	0.0009
0.8	0.0261	0.0116	0.0280	0.0082	0.0377	0.0362	0.0015
1	0.0262	0.0116	0.0262	0.0088	0.0378	0.0350	0.0027

Table 3.4(and Figure 3.4 for graph) ascertains that equation (2.7) fails except for the independence case, corresponding to $\rho = 0$ in the Gaussian copula. In particular, Figure 3.4 illustrates that the difference between left and right hand sides of (2.7) increases rapidly as the correlation gets close to 1. As equation (2.7) fails to hold under dependence, one needs to find an alternative way to evaluate the left side of

Table 3.5: Verification of equation (3.10) under general condition

ρ	(A) $\mathbb{E}[v^{T^m_{40}} \vert T^f_{40} > 0]$	(B) $\mathbb{E}[v^{T^f_{40}} \vert T^m_{40} > 0]$	(C) $\overline{A}_{40:40}$	(D) $\overline{A}_{\overline{40:40}}$	(E)=(A)+(B)	(F)=(C)+(D)	(E) - (F)
-1	0.0262	0.0117	0.0357	0.0023	0.0379	0.0379	0
-0.8	0.0260	0.0117	0.0352	0.0025	0.0377	0.0377	0
-0.6	0.0260	0.0117	0.0349	0.0028	0.0377	0.0377	0
-0.4	0.0260	0.0116	0.0345	0.0031	0.0376	0.0376	0
-0.2	0.0263	0.0116	0.0343	0.0036	0.0379	0.0379	0
0	0.0263	0.0115	0.0337	0.0041	0.0378	0.0378	0
0.2	0.0261	0.0114	0.0327	0.0048	0.0375	0.0375	0
0.4	0.0259	0.0113	0.0315	0.0057	0.0372	0.0372	0
0.6	0.0255	0.0109	0.0297	0.0067	0.0364	0.0364	0
0.8	0.0259	0.0104	0.0281	0.0082	0.0363	0.0363	0
1	0.0262	0.0089	0.0262	0.0089	0.0350	0.0350	0

(2.7). For this, we first derive the following identity relationship from equation (2.11):

$$v^{T_{xy}} + v^{T_{\overline{xy}}} = v^{T_x}|T_y > 0 + v^{T_y}|T_x > 0. \tag{3.9}$$

Then we get the expected present value by taking expectation to the both sides of the equation as

$$\overline{A}_{xy} + \overline{A}_{\overline{xy}} = \mathbb{E}[v^{T_x}|T_y > 0] + \mathbb{E}[v^{T_y}|T_x > 0]. \tag{3.10}$$

In Table 3.5, the values of equation (3.10) for various correlations are presented. Under the independent assumption, the equation (3.10) turns into the equation (2.7), because the life status of a spouse does not affect the calculation of the expected present value.

Chapter 4

Analysis of Premiums in Multiple life insurance

4.1 Analysis of Premiums under independent assumption

4.1.1 Analysis of premiums of the multiple life insurance

From the definition of random variables T_{xy} and $T_{\overline{xy}}$, we have a well-known identity

$$v^{T_{xy}} + v^{T_{\overline{xy}}} = v^{T_x} + v^{T_y}, \quad where\ T_x > 0,\ T_y > 0. \tag{4.1}$$

v^{T_x} is the present value of benefit of 1 which is payable immediately on death of a life aged x. To define both T_{xy} and $T_{\overline{xy}}$, we assume all of the insureds are alive at issue, so the conditions of Equation (4.1) should be added. If T_x and T_y are fixed, the present value of total benefits should be equal to the sum of benefits payable at the death of each insureds, taking expecations of Equation (4.1) gives us the following equation.

$$E\left[v^{T_{xy}}\right] + E\left[v^{T_{\overline{xy}}}\right] = E\left[v^{T_x}|T_y > 0\right] + E\left[v^{T_y}|T_x > 0\right] \tag{4.2}$$

Assuming the future lifetimes of the insureds are independent, we can ignore the given conditions of expections in Equation (4.2) and rewrite it using Actuarial notations as

$$\bar{A}_{xy} + \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y. \tag{4.3}$$

$\overline{A}_x$ is used to denote the net single premium of the whole life insurance with benefit of \$1 is payable immediately on death of aged x. In other words, $\overline{A}_x$ is the expected value of the present random variable for the future cash flow \$1 which is payable at T_x, and this can be expressed as

$$\overline{A}_x = E\left[v^{T_x}\right]. \tag{4.4}$$

In the spousal mortality context, $\overline{A}_{xy}$ refers to a continuous joint life insurance that represents the expected present value of the whole life insurance with benefit of \$1 which is payable immediately on the first death of couple. This can be written as

$$\overline{A}_{xy} = E\left[v^{T_{xy}}\right]. \tag{4.5}$$

A continuous last survivor life insurance is defined by $\overline{A}_{\overline{xy}}$, in contrast to $\overline{A}_{xy}$, its benefit is paid on the last death of couple.

$$\overline{A}_{\overline{xy}} = E\left[v^{T_{\overline{xy}}}\right]. \tag{4.6}$$

When lifetimes are independent, therefore, the value of last survivor life insurance can be easily calculated from Equation (4.3) given that we have already calculated the corresponding other values of the equation. For further information about Equation (4.3), see Youn et al. (2002).

The net single premium of joint life insurance is an expected value. As T_{xy} has probability density function $f_{T_{xy}}(t) = {}_tp_{xy}\,\mu_{xy}(t)$, from the definition of an expected value, we have

$$\bar{A}_{xy} = \int_0^\infty e^{-\delta t}{}_tp_{xy}\mu_{xy}(t)dt. \tag{4.7}$$

Under the indepedent assumption, Equation (4.7) becomes

$$\bar{A}_{xy} = \int_0^\infty e^{-\delta t}{}_tp_{x}{}_tp_y\,(\mu_x(t)+\mu_y(t))dt. \tag{4.8}$$

On the other hand, the net single premium of last survivor insurance can be represented as

$$\begin{aligned}\bar{A}_{\overline{xy}} &= \int_0^\infty e^{-\delta t}{}_tp_{\overline{xy}}\mu_{\overline{xy}}(t)dt \\ &= \int_0^\infty e^{-\delta t}{}_tp_x\mu_x(t)dt + \int_0^\infty e^{-\delta t}{}_tp_y\mu_y(t)dt - \int_0^\infty e^{-\delta t}{}_tp_{xy}\mu_{xy}(t)dt\end{aligned} \tag{4.9}$$

Based on Equation (4.3), the last equality of Equation (4.9) holds and it facilitates valuation of last survivor insurance when the lifetimes are independent.

Figure 4.1 illustrates the changes of premiums of multiple life insurance by the ages of insureds. As shown on the left side of Figure 4.1, the value of the joint life insurance tends to increase as the insureds get older. Since the benefit is paid when the first death occurs, much older spouse will boost the premium of such contract.

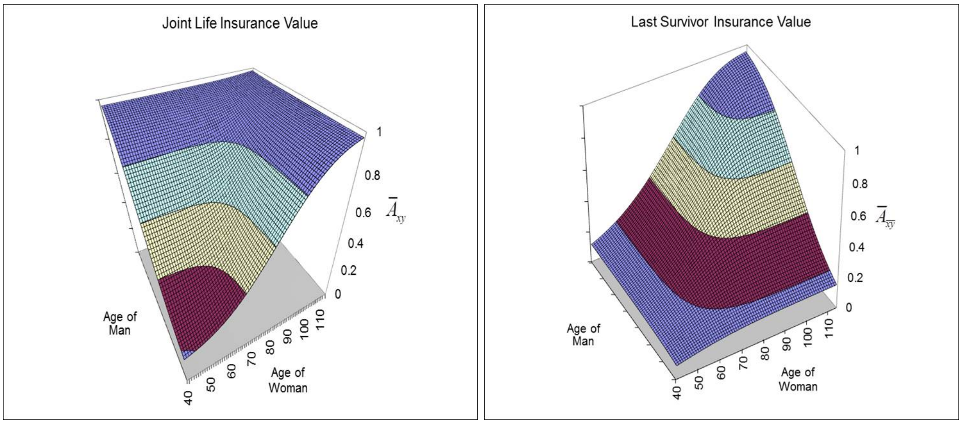


Figure 4.1: Premiums of multiple life insurance by couple's age

On the other hand, the premium of last survivor life insurance noticeably rises only if ages of both insureds rise because the benefit is paid on second death. Repeated Simpson's rule, a kind of the numerical integration methods, is used to evaluate the premiums in this section. See, e.g., **?** for methods of numerical integration.

4.1.2 Analysis of premiums of the multiple life annuity

Let us take a look at the premiums of multiple life annuities. There are two kinds of annuities, certain and life. While certain annuities guarantee fixed payments for certain amount of time, life annuities will pay certain amount of money for the lifetime of insureds. In other words, the payment period of a life annuity is a random variable as it depends on the future lifetime of insured. In the spousal motality context, T_{xy} and $T_{\overline{xy}}$ are candidates for the payment period. Whole life continuous annuity on joint life status refers to an annuity under which payments are continuously made during the future lifetime of the first death of insureds. The present value of this annuity payments is a random variable, which can be expressed as

$$\bar{a}_{\overline{T_{xy}}|} = \frac{\left(1 - e^{-\delta T_{xy}}\right)}{\delta}, \ T_{xy} > 0. \tag{4.10}$$

Taking both side of Equation (4.10), we can derive the expected present value of the joint annuity payments, $\bar{a}_{xy}$.

$$\bar{a}_{xy} = E\left[\bar{a}_{\overline{T_{xy}}|}\right] = \int_0^\infty \frac{\left(1 - e^{-\delta T_{xy}}\right)}{\delta} \ {}_tp_{xy}\mu_{xy}(t)dt \tag{4.11}$$

By the definition of $\overline{A}_{xy}$, Equation (4.7), we can express the relationship between the expected present value of joint life insurance and that of joint life annuity as below.

$$\bar{a}_{xy} = \frac{\left(1 - \bar{A}_{xy}\right)}{\delta}. \tag{4.12}$$

Similary, the expected present value of last survivor life annuity is represented using the expected present value of last survivor life insurance.

$$\bar{a}_{\overline{xy}} = \frac{\left(1 - \bar{A}_{\overline{xy}}\right)}{\delta}. \tag{4.13}$$

Figure 4.2 depicts the values of joint life annuity and last survivor life annuity based on the ages of insureds. As shown in the left side of Figure 4.2, the values of joint life annuity increase as both male and female get younger because the expected future lifetime and age is negatively correlated. If any of the insureds is close enough to w, omega, however, the value of $\overline{A}_{xy}$ in Equation (4.12) approaches 1, causing the value of joint life insurance to significuntly decrease. On the other hand, the value of last survivor annuity increases if any of the insureds is relatively young, because the future lifetime of the younger insured affects the payment period of this annuity. The

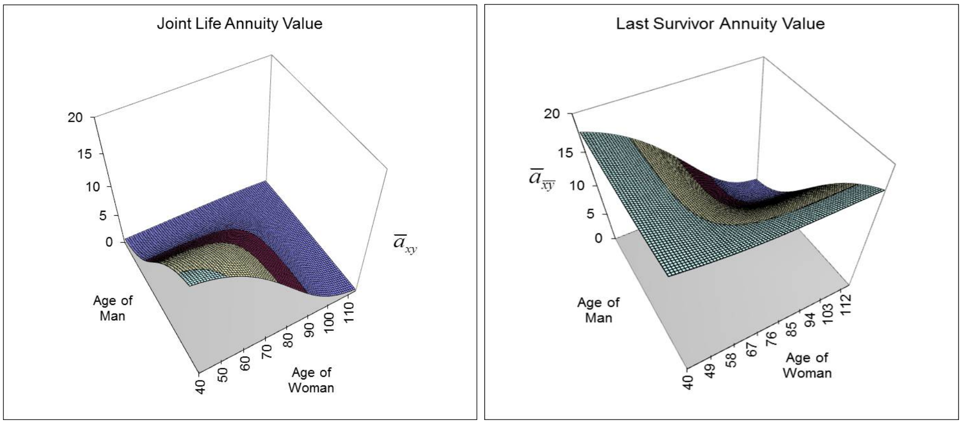


Figure 4.2: Premiums of multiple life annuity by couple's age

right panel of Figure 4.2 illustrates this tendency well. In the next section, we will evaluate premiums under dependent assumption using Gaussian copula and compare them with the corresponding premiums which assumed independency lifetimes of insureds.

4.2 Analysis of Premiums under dependent assumption

4.2.1 Premiums at various dependence levels

To analyze the effects of dependency of lifetimes on the premiums, other conditions should be fixed. We assume that the force of interest δ is 0.05, and ages of male and female are 40. Figure 4.3 shows the changes of the premiums of multiple life insur-aances by the dependent parameter ρ. We adopt the ratio of the dependent premium to the correponding independent premium to quantify the effect of depedency. In the left side of Figure 4.3, the premium ratio of joint life insurance decreases from 100% to 86% as the dependency of lifetimes increases. This tendency coincides with the shifting of the expected future lifetime of the first live to die in Table 3.3. On the

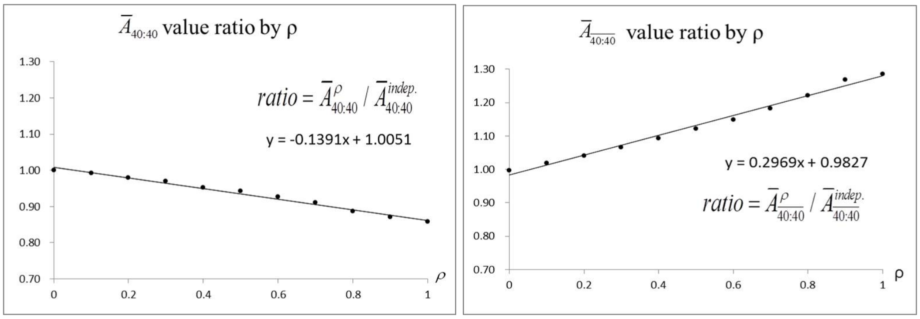


Figure 4.3: Value ratio of life insurance by degree of correlation

Table 4.1: Ratios of the multiple life insurance values by dependent parameter ρ

ρ	0	0.2	0.4	0.6	0.8	1
ratio of $\overline{A}_{xy}$	1.00	0.98	0.95	0.93	0.89	0.86
ratio of $\overline{A}_{\overline{xy}}$	1.00	1.04	1.09	1.15	1.22	1.28

contrary, the ratio of the premium of last survivor insurance increases from 100% to 128% when the correlation between the insureds increases. When we assume the dependency gets stronger, the expected time of benefit payment gets expedited, which leads to increase in expected present value of the benefit. For this reason, if insurance companies assume the lifetimes of the couples to be independent when pricing the last survivor life insurance products, insurance companies may be exposed to the risk of underestimated premiums. As shown in Table 4.2, regression analyses are used to statistically verify the relation between the dependency of lifetimes, predictor variable in the regression, and changes of multiple life insurance premiums, response varibles.

Frees et al. (1996) estimates Spearman's correlation from large contracts data of Candian insurer. The 95 percent confidence interval of the correlation coefficient is (0.41, 0.56). Applying this result to our case, the premium ratio of joint life insurance

Table 4.2: Linear regression result : Insurance value ratio v.s. Dependent parameter ρ

$\overline{A}^{\rho}_{xy}/\overline{A}^{\rho=0}_{xy}$	Coefficient	s.e.	t-test	P-value
Intercept	1.008957	0.003131	322.2767	< 0.0001
Slope	-0.14732	0.005292	-27.8382	< 0.0001
$\overline{A}^{\rho}_{\overline{xy}}/\overline{A}^{\rho=0}_{\overline{xy}}$	Coefficient	s.e.	t-test	P-value
Intercept	0.982736	0.00604	162.7056	< 0.0001
Slope	0.296919	0.010209	29.08287	< 0.0001

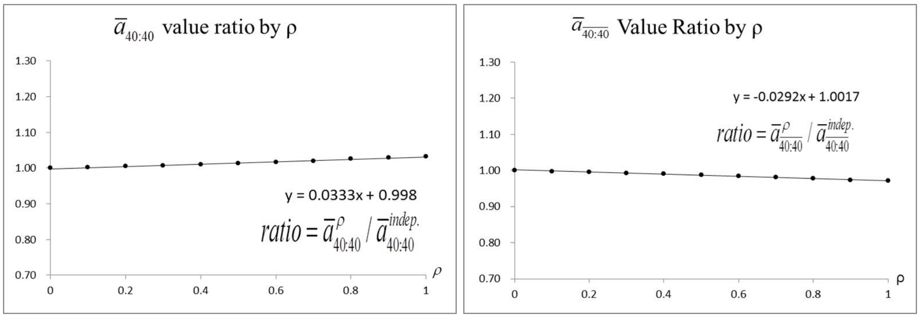


Figure 4.4: Value ratio of life annuity by degree of correlation

is about 94% and the value ratio of last survivor life insurance is about 112%. Based on this fact, when insurance companies price multiple life insurance products, they should consider the effects of dependency between the insured couples.

Let us take a look at changes of the expected present values of multiple life annuities under dependent lifetimes assumption. Compared with multiple life insurance cases, premium ratios of multiple life annuities is not as sensitive. Since the value of multiple life annuities are consist of payments which are paid at early period of the contract, shifted terminal annuity point by the dependency do not seriously contribute to change the value ratios. To be specific, discount effects of interest rate offsets the effects of shifted terminal annuity point on the contract value. As shown in Table 4.4, regression analyses are used to statistically verify the relation between the depen-

Table 4.3: Ratios of multiple life annuity values by dependent parameter ρ

ρ	0	0.2	0.4	0.6	0.8	1
$\overline{a}_{xy}$ ratio	1.00	1.00	1.01	1.02	1.03	1.03
$\overline{a}_{\overline{xy}}$ ratio	1.00	1.00	0.99	0.99	0.98	0.97

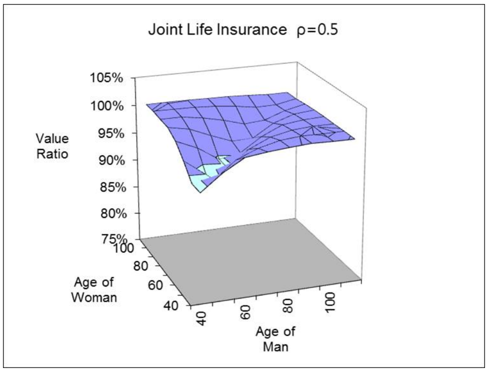
Table 4.4: Linear regression result : Annuity value ratio v.s. Dependent parameter ρ

$\overline{a}_{xy}^{\rho}/\overline{a}_{xy}^{\rho=0}$	Coefficient	s.e.	t-test	p-value
Intercept	0.997977	0.000707	1411.155	< 0.0001
Slope	0.033278	0.001195	27.83818	< 0.0001
$\overline{a}_{\overline{xy}}^{\rho}/\overline{a}_{\overline{xy}}^{\rho=0}$	Coefficient	s.e.	t-test	p-value
Intercept	1.001697	0.000594	1687.336	< 0.0001
Slope	-0.02918	0.001003	-29.0829	< 0.0001

dency of lifetimes, predictor variable in the regression, and changes of multiple life insurance premiums, response varibles. We have analyzed effects of the dependency between future lifetimes of insureds on the expected present values of multiple life insurances and annuities. These four components become building blocks of more complicated insurance products. Therefore, insurance companies should reflect the effects of dependency of insured when they price their products.

4.2.2 Impacts of ages of insureds on dependent assumption

In the previous section, the ages of insureds were fixed. We can also thick of a case with fixed dependency and varying age of the insureds. To analyze the impacts of ages of insureds, we calculate the values of multiple life insurances, reflecting age span from 40 to 110. The value ratio from previous section is used to measure the impacts of ages on the values once again. As shown in Figure 4.5 and Figure 4.6, the couple with smaller age difference is more susceptible to the dependency between lifetimes than the couple with larger age gap. Usual age difference of many married



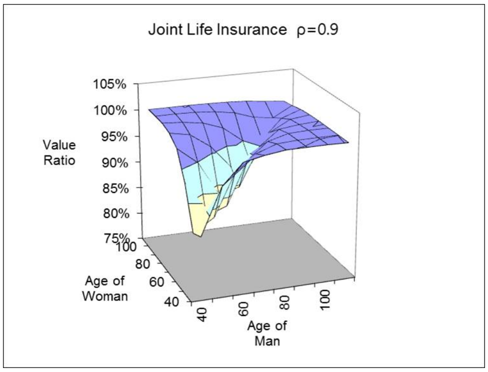


Figure 4.5: Premium of joint life insurance by ages and degree of correlation

Table 4.5: Premiums of joint life insurance by ages of the insureds and the dependency

		$\overline{A}_{xy}^{\rho=0.5}/\overline{A}_{xy}^{\rho=0}$			
x y	40	50	60	70	80
40	94%	94%	97%	99%	100%
50	97%	94%	94%	97%	99%
60	99%	97%	94%	95%	97%
70	100%	99%	97%	95%	96%
80	100%	100%	99%	98%	96%

		$\overline{A}_{xy}^{\rho=0.9}/\overline{A}_{xy}^{\rho=0}$			
x y	40	50	60	70	80
40	87%	86%	93%	98%	99%
50	95%	88%	87%	95%	99%
60	99%	95%	88%	89%	96%
70	99%	98%	96%	90%	92%
80	100%	100%	99%	96%	92%

couple is less than 10-years, which implies that a lot of multiple life insurance contracts are relatively sensitive to the dependency, causing many insurers to be exposed to dependency risk.

We already showed that the expected present value of joint life insurance decreases as the dependency between lifetimes increases. From Table 4.5, you can observe the sensitive change when the age gap is small and relatively small change when the couple's age gap is big. Also, premiums for the younger couples are more susceptible to the dependency effect than the older group.

On the contrary to the joint life insurance, the dependency of the insured couple and the expected present value of last survivor insurance are positively correlated.

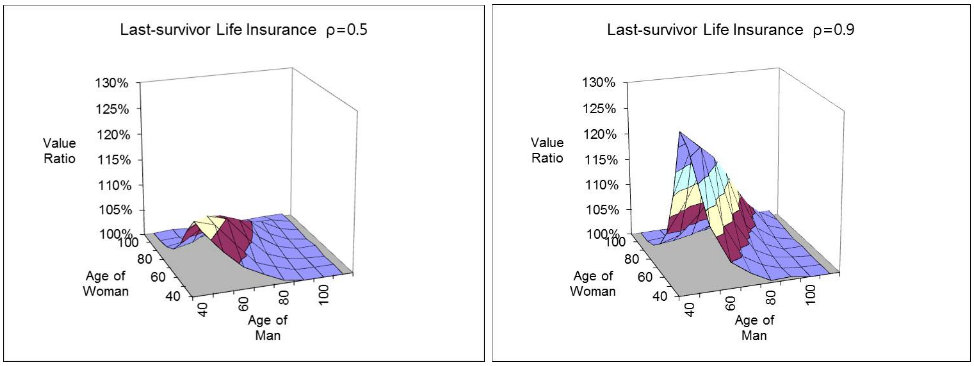


Figure 4.6: Premium of last survivor life insurance by ages of the insureds and degree of correlation

Table 4.6: Premiums of last survivor life insurance by ages of the insureds and degree of correlation

	$\overline{A}^{\rho=0.5}_{xy}/\overline{A}^{\rho=0}_{xy}$				
x y	40	50	60	70	80
40	112%	112%	109%	104%	102%
50	108%	111%	111%	107%	103%
60	104%	107%	110%	109%	105%
70	102%	103%	106%	108%	106%
80	101%	101%	102%	104%	105%

	$\overline{A}^{\rho=0.9}_{xy}/\overline{A}^{\rho=0}_{xy}$				
x y	40	50	60	70	80
40	127%	128%	115%	107%	103%
50	112%	105%	125%	112%	104%
60	105%	111%	121%	119%	108%
70	102%	104%	109%	116%	113%
80	110%	102%	103%	106%	111%

Table 4.6 presents our sample insured couples. The insured couples with small age gap are tends to be susceptible to the dependency effects, boosting the values of last survivor insurance up. The rise of the values of last survivor insurance is a serious problem that cannot be ignored for the insurer, because it implies that the current premium of the last survivor insurance is less than the true premium what they should receive.

In the previous section, the expected present value of joint life annuity increase, as the dependency of the insureds rises. Moreover, we have concluded that the multiple life annuities are more robust to the dependency when the ages of the insureds are

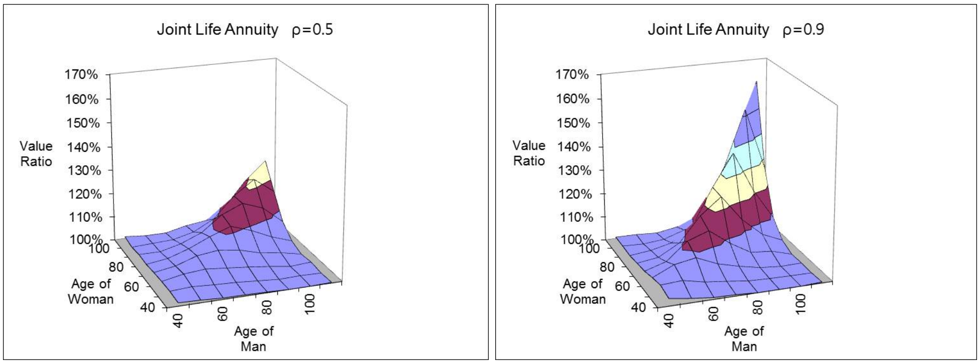


Figure 4.7: Premium of joint life annuity by age difference and degree of correlation

Table 4.7: Premiums of joint life annuity by ages of the insureds and degree of correlation

	$\overline{a}_{xy}^{\rho=0.5} / \overline{a}_{xy}^{\rho=0}$				
x y	40	50	60	70	80
40	101%	102%	102%	101%	101%
50	101%	102%	103%	103%	102%
60	101%	102%	104%	106%	105%
70	101%	101%	103%	107%	109%
80	100%	101%	102%	106%	112%

	$\overline{a}_{xy}^{\rho=0.9} / \overline{a}_{xy}^{\rho=0}$				
x y	40	50	60	70	80
40	103%	104%	103%	102%	101%
50	102%	105%	107%	104%	102%
60	101%	103%	109%	112%	107%
70	101%	102%	105%	115%	117%
80	100%	101%	103%	108%	124%

fixed at 40. When we calculate the values of life annuities as expanding the age range, we found that dependent sensitive age zone of life annuities is different from that of life insurance. In the multiple life insurances contexts, the younger aged couples are more susceptible to the dependency effect than the older group. However, when it comes to the multiple life annuities, the older groups are more susceptible to the dependency effect than the younger groups.

As shown in Table 4.7, when the ages of the insureds are 40, the premium ratio of the joint life annuity increases from 101% with $\rho = 0.5$ to 103% with $\rho = 0.9$ while that increases from 112% to 124% with the same corresponding dependency levels except the insured ages with 80. Considering the fact that many insurance policies are

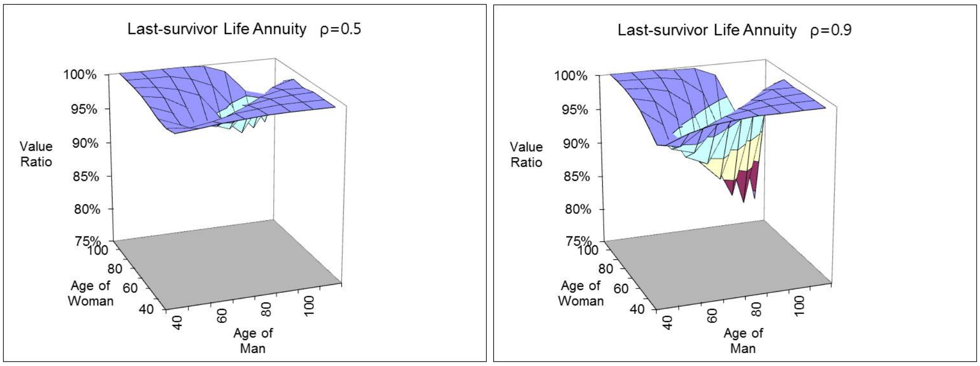


Figure 4.8: Premium of last survivor annuity by age difference and degree of correlation

Table 4.8: Premiums of last survivor life annuity by ages of the insureds and degree of correlation

	$\overline{a}_{xy}^{\rho=0.5} / \overline{a}_{xy}^{\rho=0}$					$\overline{a}_{xy}^{\rho=0.9} / \overline{a}_{xy}^{\rho=0}$				
x y	40	50	60	70	80	40	50	60	70	80
40	99%	98%	99%	99%	100%	97%	96%	98%	99%	99%
50	99%	98%	97%	98%	99%	99%	96%	94%	97%	99%
60	99%	99%	97%	96%	98%	99%	98%	93%	92%	96%
70	100%	99%	98%	96%	95%	100%	99%	97%	90%	90%
80	100%	100%	99%	97%	94%	100%	100%	99%	96%	87%

purchased by the insureds aged from 40 to 60, the expected present values of multiple life annuities are not relatively susceptible to the dependency effect. The reason why the joint annuity values of the youger insureds are robust to the dependency is that the younger insureds have more lower probability of death than the older insured even if the dependency exists, and this implies average terminal payment point is quite long. Since the long term between issue point of annuity and the terminal point makes the joint life annuity value huge, the dependency effect on the annuity issued to the young insureds is outweighed by a significant annuity value.

The expected present value of the last survivor annuity, which is payable continuously at a rate of 1 per year as long as the second to die survives, decreases as the dependency of the insureds increases. As shown in Figure 4.8, compared with the last survivor annuities issued to the younger insureds, the annuities issued to the older insureds are more sensitive to the dependency effect. We have already shown that the expected value of $T_{\overline{xy}}$ decreases as the dependent level is elevated. Although both the terminal payment points of annuities issued to the younger and the older are affected by the dependency, different impacts of the shifted terminal payment points on the each group make the last survivor annuity issued to the older insureds distingush from that of the yourger insureds.

In this section, we have analyzed that the relation between ages of insureds and premiums of multiple life insurances when the dependency levels are fixed. Note that the premiums of multiple life insurances are changed by not only the ages of insureds but also the age difference of the insureds. Therefore, the insurer can make a premium adjustment coefficient which depends on the age difference of the insureds to calculate premiums of multiple life insurances.

Chapter 5

Application premium analysis to more complicated products

Insurance companies have developed various complex insurance products with basic insurance components we have already analyzed. Premiums of some complicated products are paid continuously during the set period rather than a sigle premium at the time of issue. Selected insurances are presented in this section to investigate effects of the depedency on the products. Applying the properties of the basic components from the previous section, we analyze three multiple life insurance products, assuming ages of the insureds and the force of interests to be fixed at 40 and 0.05 repectively. Figure 5.1 and Table 5.1 present the premium ratio of the analyed products. The equivalence principle is used to calculate premiums satisfying the equality between the expected present values of future income and that of future loss.

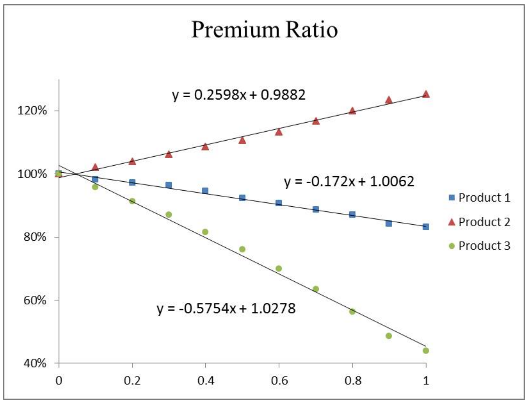


Figure 5.1: Product premium ratio by degree of correlation of T_x and T_y

Table 5.1: Premium(P_ρ) and Premium ratio($P_\rho/P_{\rho=0}$)

	0.0	0.2	0.4	0.6	0.8	1
Product 1	0.0113	0.0110	0.0107	0.0103	0.0099	0.0094
Ratio	100%	97%	94%	91%	87%	83%
Product 2	0.0055	0.0057	0.0059	0.0062	0.0066	0.0068
Ratio	100%	104%	109%	113%	120%	125%
Product 3	0.1167	0.1065	0.0953	0.0817	0.0658	0.0513
Ratio	100%	91%	82%	70%	56%	44%

5.1 Product 1

Product 1 which pays the benefit of \$1 payable immediately at the first death and collects premiums until the first death occurs is issued to the couple aged x and y. Figure 7.1 presents the future cash flows of Product 1. The distribution of random variable T_{xy} is important factor to calculate the premium of Product 1 because the time of death benefit payment and premium payment period depend on the random variable T_{xy}. Based on the properties of $\overline{A}_{xy}$ and $\overline{a}_{xy}$ from previous chapter, we can predict the premium of Product 1 to decrease as the dependency of the insureds increases. Since the premium formula of Product 1 depends on $\overline{A}_{xy}$ and $\overline{a}_{xy}$. As

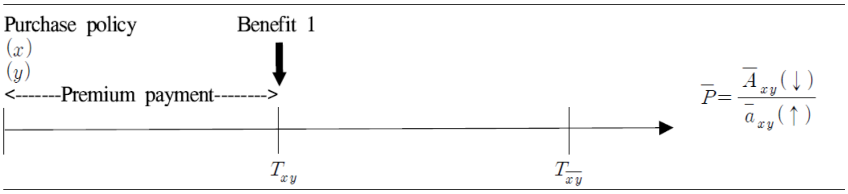


Figure 5.2: Cash flow and premium formula for Product 1

shown in the premium formula on the right side of Figure 7.1, the numerator, $\overline{A}_{xy}$, decreases at higher dependency level and the denominator, $\bar{a}_{xy}$, increases. We can also interpret changes of the premium of Product 1 intuitivly using the properties T_{xy} and $T_{\overline{xy}}$ from Table 3.3. Increasing expected value of T_{xy} implies that the delayed time of the first death due to the dependency, and as a resurlt premium payment period extends. Premium payment preriod extension can be assumed if the denominator of the formula increases. Table 5.1 shows that the premium decreases as the dependency of the insureds increases.

5.2 Product 2

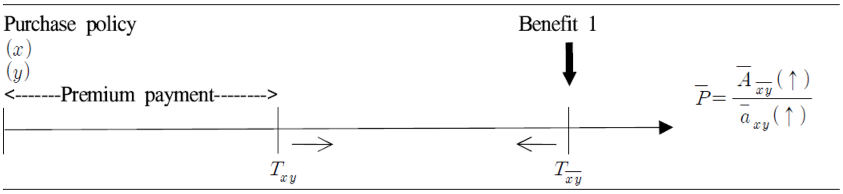


Figure 5.3: Cash flow and premium formula for Product 2

Product 2 is issued to the couple aged x and y, it pays benefit of 1$ payable immediately at the second death, and premiums are payable until the first death occurs. Figure 7.3 presents the future cash flows of Product 2. On the contrary to the previous case, the premium of Product 2 increases as the dependency of two lifetimes increases. Increasing premium of Product 2 is caused by increasing values of both $\overline{A}_{xy}$ and $\overline{a}_{xy}$. Although both denominator and numerator increase under the dependent assumption, the level of changes are different. We have already shown that the expected present values of multiple life annuities are not as susceptible as that of multiple life insurances in Figure 4.3 and Figure 4.4. The greater changes of the expected present value of joint life insurance enables the contribution of denominator to be ignored, determining the increasing tendency of the premium. Table 5.1 shows that the premium increases as the dependency of the insureds increases.

5.3 Product 3

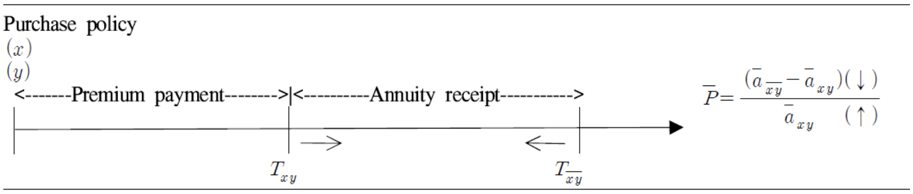


Figure 5.4: Cash flow and premium formula for Product 3

Product 3, which pays annuity of \$1 per year continuously from the first death to the last death, and collects premiums until the first death occurs, is issued to the couple aged x and y. Figure 7.5 presents the future cash flows of Product 3. As shown in Table 5.1, the premium of Product 3 significantly decreases as the dependency of two lifetimes increases. For instance, when $\rho = 0.4$, premium decreases by 20% compared to independent assumption. The sensitiveness of Product 3 is attributed to the change of annuity payment periods affected by the dependency of the insureds. As the dependent level of the insureds increases, the expected start point of payment

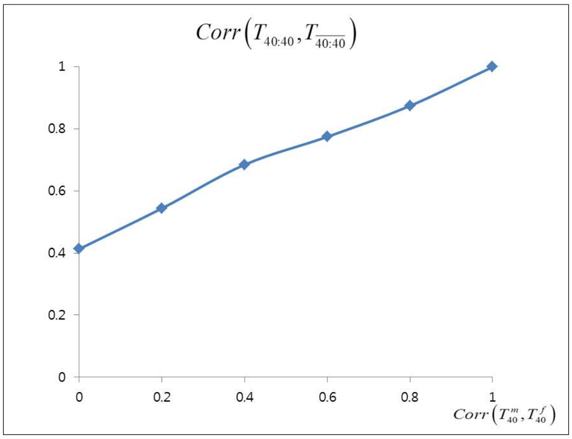


Figure 5.5: Degree of correlation of T_{xy} and $T_{\overline{xy}}$ by degree of correlation of T_x and T_y

period is delayed while the expected end point is moved forward. In other words, reduction of the expected difference between T_{xy} and $T_{\overline{xy}}$ extends the premium payment. Based on these facts, the premium formula on the right side of Figure 7.5 tells us that the premium will increase as the dependency of the insureds increases.

Decreasing tendency of Product 3 can be explained using the relationship of between single lifetime random variables and multiple lifetimes random variables. To analyze the relationship, we calculate the correlation of T_{xy} and $T_{\overline{xy}}$ using the samples from Figure 3.3 by the correlation of T_x and T_y. As shown in Figure 5.5, the correlation of multiple lifetimes random variables increases as the dependency of the insureds increases. Note that even though T_x and T_y are independent, T_{xy} and $T_{\overline{xy}}$ are not independent. However, perfect correlation between T_x and T_y does not mean that the difference between T_{xy} and $T_{\overline{xy}}$ is zero. This is reconfirmed by the fact that the premium of Product 3 is reduced utill the point that 44% of the independent premium in Table 5.1.

Chapter 6

Analysis of Reserves in Multiple life insurance

6.1 Analysis of Reserves under independent assumption

6.1.1 Relationship between Loss random variable and Reserves

For a life insurance contract issued at time 0, its reserve is generally defined by the conditional expectation of loss random variable at time $t > 0$ given that the insured is alive at that time. We denote a loss random variable at time t by ${}_tL$, which represents the difference between the present value of future benefit the insured will receive and the present value of the future premiums the insured will pay. For a contract issued to a single life aged x, the reserve at time t is therefore defined as

$$ {}_tV = \mathbb{E}[{}_tL | T_x > t]. \tag{6.1} $$

However, things are slightly different for the reserve of multiple life case. Let us assume that two people (or a couple) bought a multiple life contract. Since there are two lives instead of one, we need to consider possible cases to determine the reserve at time t, with the first case being both (x) and (y) alive, and second being only one of the two alive. The loss random variable at time t then can be written using the standrd actuarial notations as follows.

$$case\ 1:\ {}_tL|T_{xy} > t, \tag{6.2}$$

$$case\ 2:\ {}_tL|T_x > t, T_y \le t, \tag{6.3}$$

$$case\ 3:\ {}_tL|T_y > t, T_x \le t. \tag{6.4}$$

Depending on the cases, the loss random variables give rise to the different formulae of reserves which will be shown in the next chapter.

6.1.2 Multiple life insurance

We start by considering the reserve value of a fully continuous joint life insurance of 1 on the aged x and y. In this contract, the benefit is paid at the first death; thus, only the case that both insured people are alive needs to be considered. The future loss random variable at time t is

$${}_tL = v^{(T_{xy}-t)}. \tag{6.5}$$

Just as in equation (6.1), the reserve at time t is the conditional expectation of loss random variable given that both insured are alive at t. That is,

$${}_tV(\overline{A}_{xy}) = \mathbb{E}[{}_tL|T_{xy} > t] = \overline{A}_{x+t,y+t}. \tag{6.6}$$

Next, we consdier the reserve of the last survivor insurance with a benefit amount 1 payable immediately on the second death between the two lives. The future loss random variable at time t is then

$$ {}_tL = v^{(T_{\overline{xy}}-t)}. \tag{6.7} $$

However, unlike the joint life insurance case considered previously, multiple cases as in (6.2) -(6.4) should be considered separately for proper reserve calculations. At the future time t, both (x) and (y) could be alive or one of them could have already died. Therefore, three cases of reserve values should be calculated:

$$ {}_tV(\overline{A}_{\overline{xy}})^{c1} = \mathbb{E}[v^{(T_{\overline{xy}}-t)}|T_{xy} > t] = \overline{A}_{\overline{x+t,y+t}}, \tag{6.8} $$

$$ {}_tV(\overline{A}_{\overline{xy}})^{c2} = \mathbb{E}[v^{(T_{\overline{xy}}-t)}|T_x > t, T_y \le t] = \mathbb{E}[v^{T_{x+t}}|T_y \le t], \tag{6.9} $$

$$ {}_tV(\overline{A}_{\overline{xy}})^{c3} = \mathbb{E}[v^{(T_{\overline{xy}}-t)}|T_y > t, T_x \le t] = \mathbb{E}[v^{T_{y+t}}|T_x \le t], \tag{6.10} $$

where superscripts of equations, $c1$, $c2$ and $c3$, represent the statuses of its loss random variable.

6.1.3 Multiple life annuity

In this subsection we consider two standard types of multiple life annuities: The joint life and the last survivor annuities. A joint life annuity pays continuously at a rate of 1 per year while both insureds are still alive. The future loss random variable at time t of this contract is

$$ {}_tL = \frac{1 - v^{(T_{xy}-t)}}{\delta}. \tag{6.11} $$

The reserve of this joint life annuity at time t is defined as the conditional expectation of (6.11) given that both x and y are still alive at time t:

$$ {}_tV\left(\bar{a}_{xy}\right) = E\left[\left.\frac{1-v^{(T_{xy}-t)}}{\delta}\right| T_{xy} > t\right] = \bar{a}_{x+t,y+t}. \tag{6.12}$$

A last survivor annuity means a life annuity pays continuously at a rate of 1 per year while at least one of the two is still alive. Similar to joint life annuity, loss random variable at time t is

$$ {}_tL = \frac{1-v^{(T_{\overline{xy}}-t)}}{\delta}. \tag{6.13}$$

Like the last survivor life insurance, the reserve of last survivor annuity is obtained separately for three cases:

$$ {}_tV\left(\bar{a}_{\overline{xy}}\right)^{c1} = E\left[\left.\frac{1-v^{(T_{\overline{xy}}-t)}}{\delta}\right| T_{xy} > t\right] = \bar{a}_{\overline{x+t,y+t}}, \tag{6.14}$$

$$ {}_tV\left(\bar{a}_{\overline{xy}}\right)^{c2} = E\left[\left.\frac{1-v^{(T_{\overline{xy}}-t)}}{\delta}\right| T_x > t, T_y \le t\right] = \frac{1-E\left[\left. v^{T_{x+t}}\right| T_y \le t\right]}{\delta}, \tag{6.15}$$

$$ {}_tV\left(\bar{a}_{\overline{xy}}\right)^{c3} = E\left[\left.\frac{1-v^{(T_{\overline{xy}}-t)}}{\delta}\right| T_y > t, T_x \le t\right] = \frac{1-E\left[\left. v^{T_{y+t}}\right| T_x \le t\right]}{\delta}, \tag{6.16}$$

where superscripts of equations, $c1$, $c2$ and $c3$, represent the statuses of its loss random variable.

6.2 Analysis of reserves under dependent assumption

6.2.1 Reserves changes when there is dependency

As we mentioned before, reserves at time t are conditional expectations. Therefore, some reserve values can be materially different when the dependency exists. In this spirit, we may consider four different situations depending on whether the dependence is present or not, and the process of interest is pricing or reserving, as shown in table 6.1. We have included the pricing dependence parameter, $\rho_{pricing}$, because the correlation also has an effect on the premium calculation, as seen in Section 2. In Table 6.1 we denote reserve of the insurance contract at time t as

$$ {}_tV\left(\rho_{reserves}, \rho_{pricing}\right). \tag{6.17}$$

In the table, for example, ${}_tV(\rho, 0)$ means that we assume dependence for the reserving, but with the premium derived from independence.

Table 6.1: Combination of the premium ρ and the reserves ρ

		Premium process	
		Indep.	Dep.
Reserves process	Indep.	${}_tV(0,0)$	${}_tV(0,\rho)$
	Dep.	${}_tV(\rho,0)$	${}_tV(\rho,\rho)$

6.2.2 Reserves of multiple life insurance: both the insured alive

The reserves of joint life insurance contracts, assuming that the insureds pay a single premium at issue and receives a benefit of 1, increases from 0.2 to 1 over time. Figure 6.1 illustrates the reserve values during the contract period. Note that there is a jump at time zero. Although the reserves are zero at time 0, determined by an equivalence

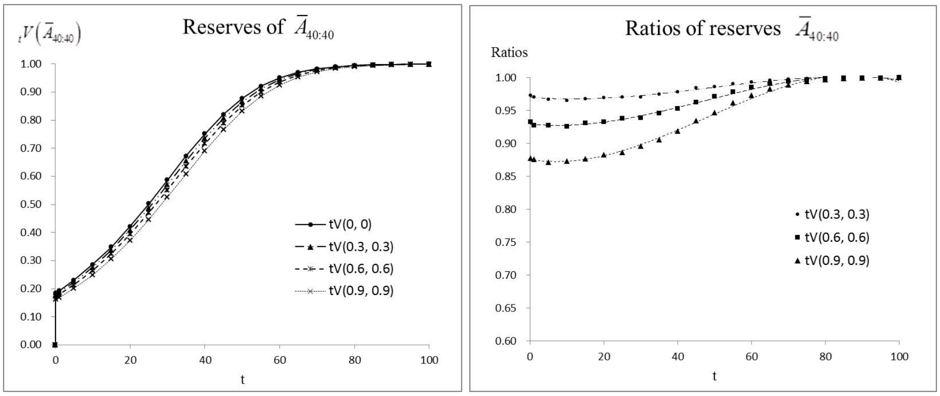


Figure 6.1: Reserves and ratio of reserves of $\overline{A}_{40:40}$ (Assumption: $T_{40} > t, T_{40} > t$)

Table 6.2: Reserves and ratio of reserves of $\overline{A}_{40:40}$ (Assumption: $T_{40} > t, T_{40} > t$)

${}_tV(\overline{A}_{40:40})$	Time	0	0.1	1	5	10	25	50	75	100
Value	$tV(0.0, 0.0)$	0	0.184	0.193	0.23	0.286	0.502	0.878	0.99	0.999
	$tV(0.3, 0.3)$	0	0.179	0.187	0.223	0.276	0.488	0.866	0.988	0.999
	$tV(0.6, 0.6)$	0	0.172	0.179	0.214	0.264	0.471	0.852	0.987	0.999
	$tV(0.9, 0.9)$	0	0.162	0.169	0.201	0.249	0.445	0.831	0.984	0.999
Ratio	$tV(0.3, 0.3)$		0.973	0.97	0.967	0.965	0.971	0.987	0.998	1
	$tV(0.6, 0.6)$		0.932	0.928	0.927	0.925	0.938	0.971	0.997	1
	$tV(0.9, 0.9)$		0.877	0.876	0.871	0.873	0.886	0.947	0.994	1

premium principle, the values jump from zero and they are affected by ρ used in reserve calculation.

As ρ increases, the reserves of joint life insurance decrease. In the left panel of Figure 6.1, however, the gaps between independent reserve and dependent reserves reduce as time passes. This tendency can be explained by two reasons. The first reason is an aging effect. The dependence effect gets gradually smaller as one gets older, because the magnitude of the mortality becomes large enough at higher ages to outweigh the dependence effect. As time passes, the insured gets older and reserve value gets closer to the independent reserve. Second, all reserves have the same

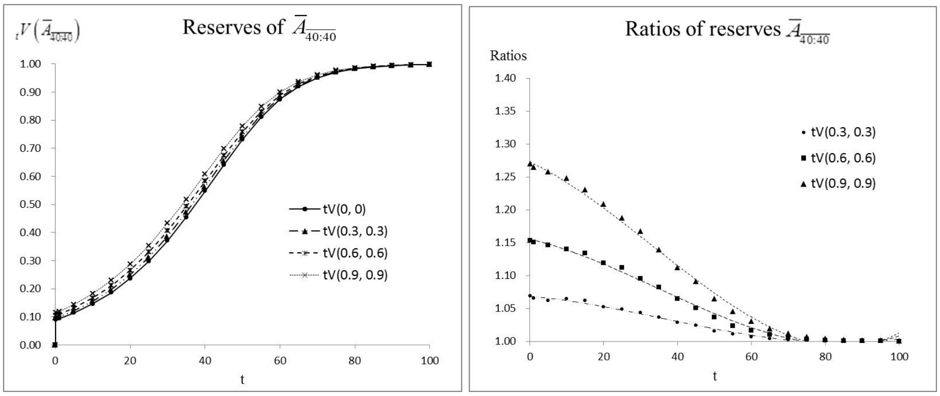


Figure 6.2: Reserves and ratio of reserves of $\overline{A}_{\overline{40:40}}$ (Assumption: $T_{40} > t, T_{40} > t$)

Table 6.3: Reserves and ratio of reserves of $\overline{A}_{\overline{40:40}}$ (Assumption: $T_{40} > t, T_{40} > t$)

$_tV(\overline{A}_{\overline{40:40}})$	Time	0	0.1	1	5	10	25	50	75	100
Value	tV(0.0, 0.0)	0	0.090	0.094	0.115	0.146	0.298	0.732	0.971	0.998
	tV(0.3, 0.3)	0	0.096	0.100	0.122	0.156	0.313	0.743	0.972	0.998
	tV(0.6, 0.6)	0	0.104	0.108	0.132	0.167	0.331	0.759	0.974	0.998
	tV(0.9, 0.9)	0	0.114	0.119	0.144	0.182	0.354	0.779	0.977	0.998
Ratio	tV(0.3, 0.3)		1.069	1.066	1.063	1.065	1.049	1.016	1.001	1
	tV(0.6, 0.6)		1.153	1.151	1.147	1.140	1.112	1.037	1.004	1
	tV(0.9, 0.9)		1.269	1.264	1.257	1.248	1.187	1.064	1.006	1

terminal value. Since joint life insurance is a type of whole life insurance contracts, its terminal value must be 1 regardless of the correlation. The right panel of Figure 6.1 shows this tendency using a ratio dependent reserves to independent reserves as its measure.

Frees et al. (1996) estimated the value 0.49 of Spearman's correlation coefficient from the multiple insurance data set. The estimate 0.49 indicates a strong statistical dependence between the husband's and wife's mortality. According to Fang et al. (2002), Spearman's correlation coefficient is almost same as the parameter of Gaussian copula. Some selected reserve values are presented in Table 6.2. In the table 6.2, note that reserves of $\rho = 0.6$ at time 10 is proportional to 92.5$\%$ compared with the value calculated under independent assumption.

On the other hand, the reserves of the last survivor insurance contracts rise from 0.1 to 1, given that the both insured people are still alive at time t. Compared with the reserves of the joint life insurance, the reserves of last survivor insurance increase as the dependence gets stronger. The left panel of Figure 6.2 shows that the more dependency the insured people have, the higher reserves should be. Since the expected value of $T_{\overline{xy}}$ goes down when the correlations increase, so the dependent reserves always lay onto the independent reserves line until the contract is expired. The right panel of Figure 6.2 demonstrates that the ratio of the corresponding dependent reserves to the independent reserve during the contract period. At the initial period of the contract, the ratio differences are big but they become smaller when the insureds get older. Again, as the terminal value of reserves should be 1, all reserve ratios converge to 1 in the long run. Some selected reserve values are presented in Table 6.3. In the Table, for example, the reserve of $\rho = 0.6$ at time 5 is proportional to 114.7$\%$ compared with the value calculated under independent assumption.

6.2.3 Reserves of multiple life annuity: all of the insured alive

A joint life annuity pays at rate 1 per year while all of the insureds are still alive, with its reserve calculated using equation (6.12). After the issue, the expected payment period shortens because the mortality increases. This means that the reserves of joint life annuity goes to zero eventually. Since the expectation of T_{xy} increases as the correlation becomes higher, starting points of the reserves around time zero are high when its correlation is high.

In the right panel of Figure 6.3, unlike the joint life insurance case, the ratios

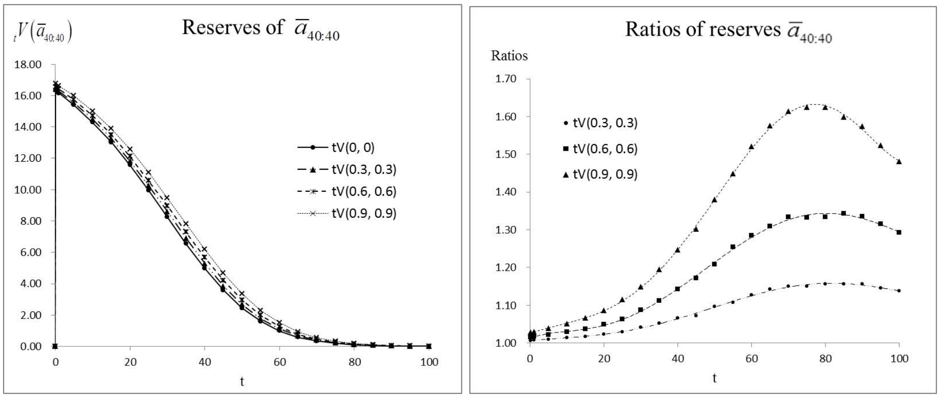


Figure 6.3: Reserves and ratio of reserves of $\overline{a}_{40:40}$ (Assumption: $T_{40} > t, T_{40} > t$)

Table 6.4: Reserves and ratio of reserves of ${}_tV(\overline{a}_{xy})$

${}_tV(\overline{a}_{40:40})$	Time	0	0.1	1	5	10	25	50	75	100
Value	$tV(0.0, 0.0)$	0	16.313	16.146	15.391	14.288	9.954	2.445	0.201	0.011
	$tV(0.3, 0.3)$	0	16.413	16.261	15.542	14.488	10.25	2.682	0.231	0.013
	$tV(0.6, 0.6)$	0	16.563	16.425	15.727	14.715	10.579	2.955	0.268	0.015
	$tV(0.9, 0.9)$	0	16.765	16.625	15.984	15.015	11.096	3.375	0.327	0.017
Ratio	tV(0.3, 0.3)	1	1.006	1.007	1.010	1.014	1.030	1.097	1.150	1.138
	$tV(0.6, 0.6)$	1	1.015	1.017	1.022	1.030	1.063	1.208	1.332	1.292
	$tV(0.9, 0.9)$	1	1.028	1.030	1.039	1.051	1.115	1.380	1.625	1.480

of reserves increase from 1 and then decrease to 1, making cancave shape in their peaks. In the early time periods, although the reserves are different, as seen in the left panel, the large size of annuity values makes the ratios differences indistinguishable in the figure. As time passes, as the annuity reserves become small, the ratios are more distinct, indicating that the amount of expected annuity value are small and the amount of differences after taking ratio can not be ignored anymore. Finally, at the terminal point of the contract, all reserves should be zero so the ratios of reserves go to 1. The right panel of Figure 6.3 shows well this tendency.

Let us take a look at the reserves of the last survivor annuity. Just like the reserves

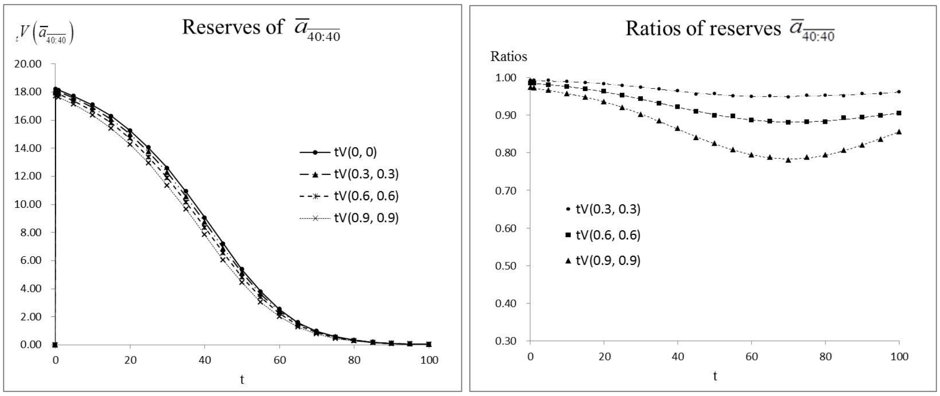


Figure 6.4: Reserves and ratio of reserves of $\overline{A}_{\overline{40:40}}$ (Assumption: $T_{40} > t, T_{40} > t$)

Table 6.5: Reserves and ratio of reserves of $_tV(\bar{a}_{\overline{xy}})$

$_tV(\bar{a}_{\overline{40:40}})$	Time	0	0.1	1	5	10	25	50	75	100
Value	$tV(0.0, 0.0)$	0	18.204	18.119	17.705	17.076	14.041	5.362	0.581	0.038
	$tV(0.3, 0.3)$	0	18.079	17.995	17.562	16.886	13.749	5.134	0.553	0.037
	$tV(0.6, 0.6)$	0	17.929	17.835	17.369	16.665	13.371	4.823	0.512	0.035
	$tV(0.9, 0.9)$	0	17.72	17.622	17.115	16.352	12.928	4.427	0.457	0.033
Ratio	tV(0.3, 0.3)	1	0.993	0.993	0.992	0.989	0.979	0.958	0.953	0.962
	$tV(0.6, 0.6)$	1	0.985	0.984	0.981	0.976	0.952	0.899	0.882	0.906
	$tV(0.9, 0.9)$	1	0.973	0.973	0.967	0.958	0.921	0.826	0.788	0.856

of joint life case, the reserves of the last survivor annuity decrease to zero over time. This is because, in the initial periods of the contract, the expected payment period is longer than that of the terminal period. However, compared with the joint life annuity case, the reserves of the last survivor annuity is getting lower as the dependence becomes stronger. It is convenient for readers to think that the difference between T_{xy} and $T_{\overline{xy}}$ becomes small as the correlation increases. For the ratios of reserves of the last survivor annuity, as seen in the right panel of Figure 6.4, they start from 1 and makes a convex shape, finally going to 1 again as time goes to infinity. Some selected numbers are presented in Table 6.5. When we analyze the reserves of the joint life insurance and annuity contracts, only one situation where both are alive needs to be considered. However, for the last survivor insurance and annuity, other situations, that is, one of the insureds already has died at the valuation time, also needs to be accounted for.

6.2.4 Reserves of multiple life insurance and annuity: one of the insureds already died

In this subsection, we consider the reserve calculation of the last survivor contracts under the assumption that one of the insureds has already died before the evaluation time t. In this case, the reserve calculation relies on the conditions like (6.3) or (6.4). Figure 6.5 shows that the reserves of last survivor insurance and its ratio under the condition (6.3). When the future lifetimes of the insureds are mutually independent, the reserves of the contract at time t, ${}_tV(0,0)$, is same as the reserves of single whole life insurance. However, the reserves change if there exists a dependency between the mortalities of the insureds. As seen in Table 6.6, when ρ is 0.3, the reserve value at time 0.5 is 0.349 which is higher than the independent counterpart 0.162. In the initial period, the starting reserves are quite different from the reserves under the condition that both insureds are alive.

To be specific, when the parameter ρ is 0.6, the reserves at time 1 is 0.649, which is higher than 0.108 in the table 6.3 which assumed that both insureds are still alive at time 1. To understand this substantial difference, let us consider a correlated pair of random variables. Under a strong dependence, one would expect that the two random variables move together; if one variable realizes a small value, so does the other variable. Therefore, when there is a high dependence between the future lifetimes of the insureds, the reserve of the last survivor insurance is close to 1 in the initial period

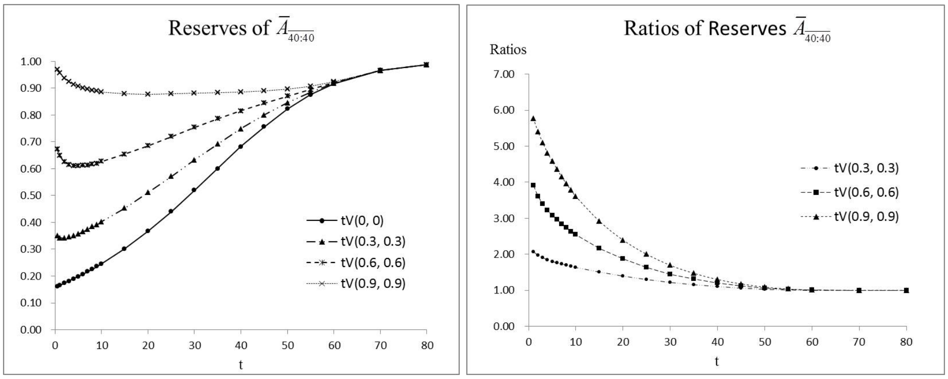


Figure 6.5: Reserves and ratios of reserves of $\overline{A}_{\overline{40:40}}$ (Assumption : $T_{40} > t, T_{40} < t$)

Table 6.6: Reserves of multiple life insurance: one of the insured already die

${}_tV(\overline{A}_{\overline{40:40}})$	Time	0.5	1	2	3	4	5	10	25	50
Value	$tV(0.0, 0.0)$	0.162	0.166	0.174	0.181	0.190	0.198	0.246	0.440	0.822
	$tV(0.3, 0.3)$	0.349	0.343	0.342	0.346	0.351	0.357	0.403	0.571	0.845
	$tV(0.6, 0.6)$	0.674	0.649	0.626	0.615	0.611	0.610	0.627	0.719	0.87
	$tV(0.9, 0.9)$	0.970	0.955	0.936	0.924	0.914	0.907	0.886	0.879	0.896
Ratio	$tV(0.3, 0.3)$	2.157	2.069	1.968	1.908	1.849	1.802	1.638	1.300	1.027
	$tV(0.6, 0.6)$	4.158	3.918	3.606	3.394	3.217	3.083	2.553	1.636	1.058
	$tV(0.9, 0.9)$	5.984	5.77	5.396	5.095	4.812	4.58	3.604	1.999	1.089

of the contract. The terminal reserves are almost same regardless of the dependency level, because the insureds are getting older and their mortalities increase, which means that correlation effect on the reserves becomes negligible.

In the case of the last survivor annuity, as presented in Table 6.7 and Figure 6.6, the starting points of reserves prone to decrease from the high value to zero as the dependency increases, because it is likely that the ramaining period of annuity becomes shorter than the independence case, given that the other insured is already dead. Opposite to the last survivor insurance case, the reserves of the last survivor annuity get smaller as the future lifetime random variables have a stronger correlation. For

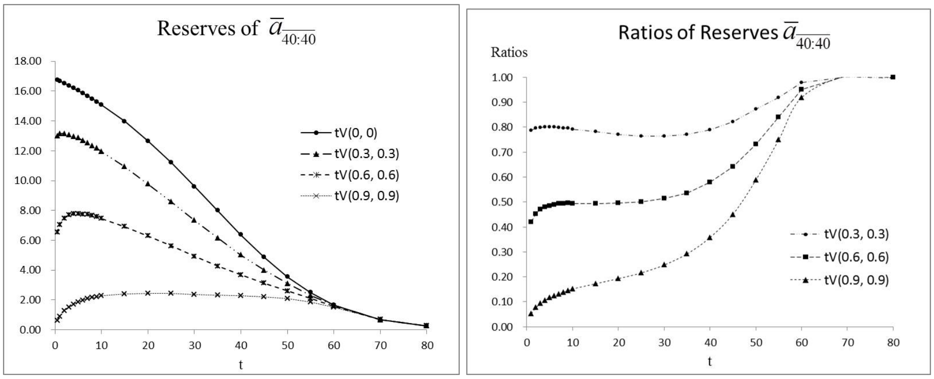


Figure 6.6: Reserves and ratios of reserves of $\overline{a}_{\overline{40:40}}$ (Assumption : $T_{40} > t, T_{40} < t$)

Table 6.7: Reserves of multiple life annuity: one of the insured already die

${}_tV(\overline{a}_{\overline{40:40}})$	Time	0.5	1	2	3	4	5	10	25	50
Value	$tV(0.0, 0.0)$	16.76	16.69	16.53	16.37	16.20	16.04	15.09	11.21	3.553
	$tV(0.3, 0.3)$	13.011	13.15	13.17	13.08	12.98	12.87	11.95	8.572	3.103
	$tV(0.6, 0.6)$	6.526	7.025	7.484	7.694	7.785	7.791	7.451	5.613	2.597
	$tV(0.9, 0.9)$	0.610	0.891	1.271	1.529	1.724	1.864	2.288	2.421	2.089
Ratio	$tV(0.3, 0.3)$	0.776	0.788	0.797	0.799	0.801	0.802	0.792	0.765	0.873
	$tV(0.6, 0.6)$	0.389	0.421	0.453	0.470	0.480	0.486	0.494	0.501	0.731
	$tV(0.9, 0.9)$	0.036	0.053	0.077	0.093	0.106	0.116	0.152	0.216	0.588

example, in the table 6.7, the reserves with $\rho = 0.3$ at time 1 is 13.146 which is lower than 16.688, the reserves value under independence; this is because if one of the two insureds has died in the initial period of the contract, the another is likely to die within a very short time.

Chapter 7

Application reserves analysis to more complicated products

The purpose of this section is to offer a reserve analysis for more complicated insurance Products which are combined by the building blocks we have already studied in Section 4. The Product assumption used in the reserves calculation is same as before: a man of aged 40, a woman of aged 40, and the force of interest is 0.05. In this section, we especially consider the case that dependence for reserveing, but with the premium derived from independence, which is denoted by ${}_tV(\rho, 0)$. We consider four different multiple-life insurance Products with various life statuses and payment patterns.

7.1 Product analysis 1

Product 1 is similar to the joint life insurance, but the insureds continuously pay premiums until the point that the first death occurs, and benefit 1 is payable at that time. The future cash flow of the analyzed Product is expressed in Figure 7.1. Using

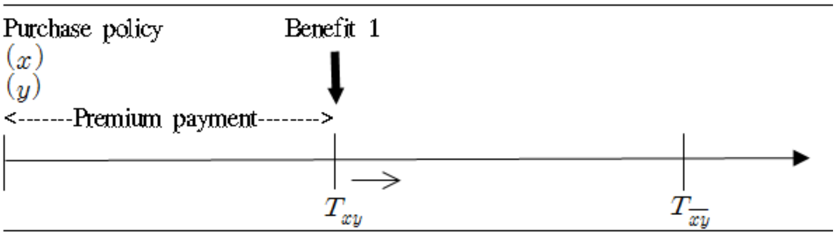


Figure 7.1: Cash flow and premium formula for Product 1

the equivalence principle of premium, under the independent assumption, the net premium at issue is given by

$$\overline{P} = \frac{\overline{A}_{xy}}{\overline{a}_{xy}}. \tag{7.1}$$

At time t, the prospective reserve, defined in (6.1), can be written as the expected value of the conditional random variable (6.2), that is,

$$_tV = \mathbb{E}[_tL | T_{xy} > t] \tag{7.2}$$

which can be explicitly written as

$$_tL = v^{(T_{xy}-t)} - \overline{P}\,\overline{a}_{\overline{T_{xy}-t}|} | T_{xy} > t. \tag{7.3}$$

Hence, for Product 1, we can express the reserves as a linear function of three components: EPV of joint life insurance, the Product premium, and EPV of joint life annuity.

$$\begin{aligned} _tV &= \mathbb{E}\Big[v^{(T_{xy}-t)} - \overline{P}\,\overline{a}_{\overline{T_{xy}-t}|} | T_{xy} > t\Big] \\ &= \overline{A}_{x+t,y+t} - \overline{P}\,\overline{a}_{x+t,y+t}. \end{aligned} \tag{7.4}$$

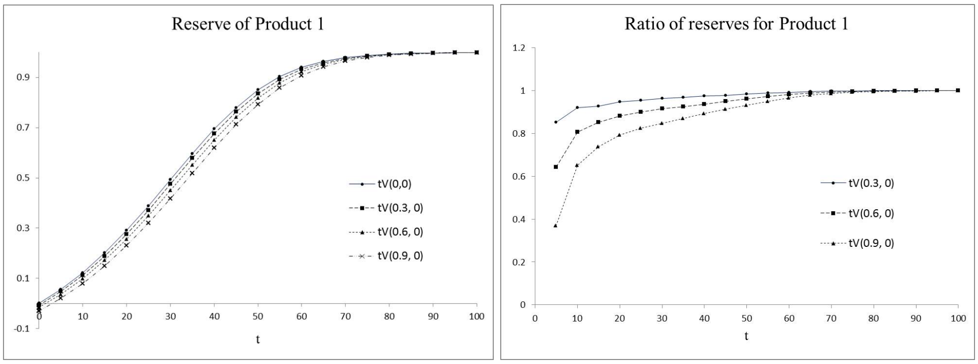


Figure 7.2: Reserves of Product 1 at time t (Assumption : $T_{40:40} > t$)

Table 7.1: Reserves of Product 1 at time t (Assumption : $T_{40:40} > t$)

	Time	0	0.1	5	10	25	50	75	100
Value	$tV(0.0,0)$	0	0.001	0.056	0.123	0.389	0.851	0.988	0.999
	$tV(0.3,0)$	-0.008	-0.007	0.048	0.113	0.371	0.836	0.986	0.999
	$tV(0.6,0)$	-0.016	-0.014	0.036	0.099	0.350	0.819	0.984	0.999
	$tV(0.9,0)$	-0.029	-0.029	0.021	0.080	0.321	0.792	0.980	0.999
Ratio	$tV(0.3,0)$		-9.004	0.852	0.919	0.954	0.983	0.998	1
	$tV(0.6,0)$		-19.389	0.644	0.806	0.900	0.962	0.996	1
	$tV(0.9,0)$		-38.263	0.369	0.651	0.825	0.932	0.992	1

The reserve values (7.4) and their selected numbers are presented in Figure 7.2 and Table 7.1, over different ρ's for both pricing and reserving sides. Figures 7.2 illustrates that as ρ gets larger, the reserves tends to go down. For example, in Table 7.1, when the independent reserve at time 5 is 0.056, the dependent reserves are 0.048, 0.036 and 0.021 when each of the correlation parameter is 0.3, 0.6, and 0.9, respectively. Note that a start point of the independent reserve is zero which is different from the reserves in, e.g., (6.8), where its loss random variable does not consider the future incomes.

Moreover, the starting values of other dependent reserves have negative values.

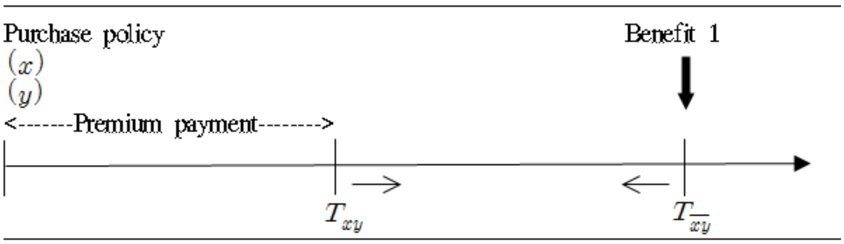


Figure 7.3: Cash flow and premium formula for Product 2

Lee et al. (2013) found that the value of $\overline{A}_{xy}$ decreases as the correlation of the insured goes up. This is because, for the joint life insurance, the premium derived from indepdence is higher than that from a positive dependence. That is, the company has received more money than needed, and this explains negative reserves for the early time periods.

7.2 Product analysis 2

Product 2 bears some resemblance to the last survivor insurance. As for Product 1, the insureds continuously pay premium until the first death occurs, but a benefit 1 will be paid at time of the second death. The future cash flow of Product 2 we consider is depicted in the figure 7.3. The equivalence principle of premium gives the net premium at issue as

$$\overline{P} = \frac{\overline{A}_{\overline{xy}}}{\overline{a}_{xy}}. \tag{7.5}$$

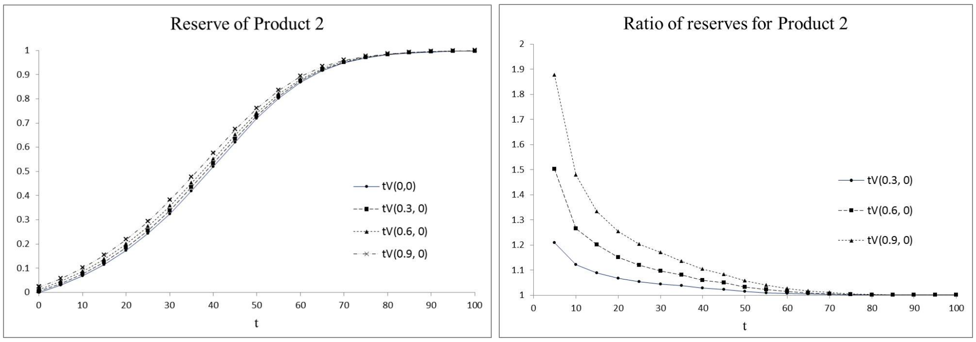


Figure 7.4: Reserves of Product 2 at time t (Assumption : $T_{40:40} > t$)

Table 7.2: Reserves of Product 2 at time t (Assumption : $T_{40:40} > t$)

	Time	0	0.1	5	10	25	50	75	100
Value	$tV(0.0,0)$	0	0	0.030	0.068	0.243	0.718	0.97	0.998
	$tV(0.3,0)$	0.005	0.006	0.036	0.076	0.256	0.730	0.971	0.998
	$tV(0.6,0)$	0.012	0.013	0.045	0.086	0.272	0.742	0.973	0.998
	$tV(0.9,0)$	0.021	0.022	0.057	0.100	0.293	0.760	0.975	0.998
Ratio	$tV(0.3,0)$		13.267	1.209	1.121	1.054	1.016	1.001	1
	$tV(0.6,0)$		29.999	1.501	1.266	1.119	1.033	1.003	1
	$tV(0.9,0)$		51.72	1.877	1.479	1.204	1.058	1.006	1

To calculate the reserve, we consider three cases written as

$$_tL = v^{(T_{\overline{xy}}-t)} - \overline{P}\,\bar{a}_{\overline{T_{xy}-t}|}|T_{xy} > t, \tag{7.6}$$

$$_tL = v^{(T_x-t)}|T_x > t, T_y \leq t, \tag{7.7}$$

$$_tL = v^{(T_y-t)}|T_y > t, T_x \leq t. \tag{7.8}$$

Figure 7.4 illustrates the case where both insureds are alive at the valuation point. When the future lifetimes of the insureds are independent, the reserve starts from zero and goes to one over time. Note that the starting reserves increase, as the $\rho_{reserves}$ increases. According to the above premium equation, the premium of Product 2

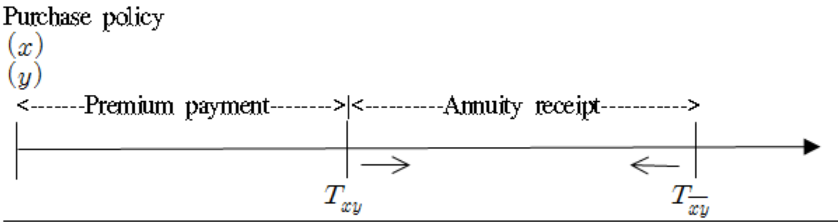


Figure 7.5: Cash flow and premium formula for Product 3

increases, as the correlation parameter has a high value. If the company receives the premium derived from independence assumption, but assumes that the future lifetimes of the insureds are correlated, the premium the company received will be inadequate. That is, the insurer loses money on this Product at issue. The other two cases (7.7) and (7.8) can be similarly inferred from Section 6.2.4, and the details are omitted.

7.3 Product analysis 3

The future cash flow of Product 3 is depicted in the Figure 7.5. In this case, the insureds pay premiums continuously until the first death, and after that, an annuity is continuously paid by insurer until the second death. From the equivalence principle of premium, the insurer's premium at issue is

$$\overline{P} = \frac{\overline{a}_{\overline{xy}} - \overline{a}_{xy}}{\overline{a}_{xy}}. \tag{7.9}$$

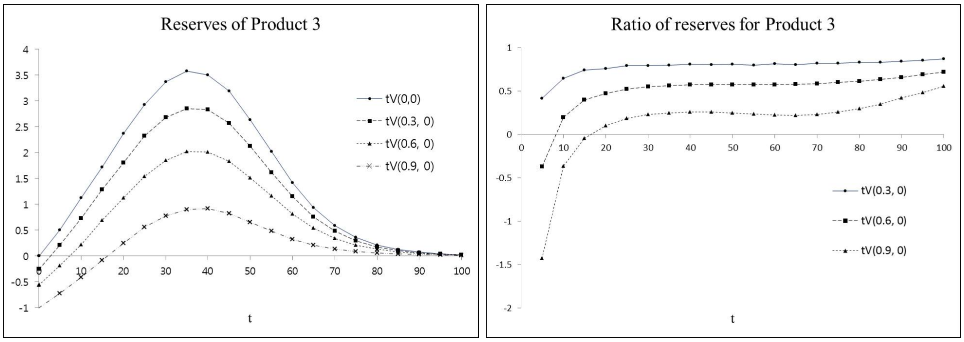


Figure 7.6: Reserves of Product 3 at time t (Assumption : $T_{40:40} > t$)

Table 7.3: Reserves of Product 3 at time t (Assumption : $T_{40:40} > t$)

	Time	0	0.1	5	10	25	50	75	100
Value	$tV(0.0, 0)$	0	0.004	0.506	1.125	2.930	2.636	0.357	0.026
	tV(0.3, 0)	-0.256	-0.254	0.212	0.727	2.323	2.130	0.294	0.023
	tV(0.6, 0)	-0.563	-0.554	-0.187	0.223	1.538	1.517	0.215	0.019
	tV(0.9, 0)	-1.006	-1.001	-0.722	-0.409	0.556	0.652	0.092	0.014
Ratio	$tV(0.3, 0)$		-68.445	0.419	0.646	0.793	0.808	0.822	0.874
	$tV(0.6, 0)$		-149.278	-0.369	0.198	0.525	0.576	0.601	0.719
	$tV(0.9, 0)$		-269.572	-1.425	-0.363	0.190	0.247	0.258	0.556

The loss random variable should be distinguished by the joint life status. The three cases can be written as

$$ {}_tL = \left(\bar{a}_{\overline{T_{\overline{xy}}-t|}} - \bar{a}_{\overline{T_{xy}-t|}}\right) - \overline{P}\bar{a}_{\overline{T_{xy}-t|}} | T_{xy} > t, \tag{7.10} $$

$$ {}_tL = \bar{a}_{\overline{T_x-t|}} | T_x > t, T_y \leq t, \tag{7.11} $$

$$ {}_tL = \bar{a}_{\overline{T_y-t|}} | T_y > t, T_x \leq t. \tag{7.12} $$

Let us analyze the first case. Figure 7.6 illustrates the reserves under the assumption that both insureds are alive at the evaluation point. The top line of the four lines in the graph means the independent reserves which calculated under the independent assumption in pricing and reserving. This independent reserve starts from zero and goes back to zero, making "concave down" shape during the Product period. Because of the premium calculated by equivalence premium principle, starting point should be zero. To understand the shape of the curve, let us state the reserves at time t given that both the insured are still alive.

$$
\begin{aligned}
{}_tV &= \mathbb{E}\Big[\Big(\overline{a}_{\overline{T_{\overline{xy}}-t}|} - \overline{a}_{\overline{T_{xy}-t}|}\Big) - \overline{P}\overline{a}_{\overline{T_{xy}-t}|} | T_{xy} > t\Big] \\
&= \big(\overline{a}_{\overline{x+t:y+t}} - \overline{a}_{x+t:y+t}\big) - \overline{P}\overline{a}_{x+t:y+t}. \qquad (7.13)
\end{aligned}
$$

For instance, let us assume that the insurer is at time 10, 40 and 60, where the reserves are given by

$$
{}_{10}V = \Big(\overline{a}_{\overline{x+10:y+10}} - \overline{a}_{x+10:y+10}\Big) - \overline{P}\overline{a}_{x+10:y+10}, \qquad (7.14)
$$

$$
{}_{40}V = \Big(\overline{a}_{\overline{x+40:y+40}} - \overline{a}_{x+40:y+40}\Big) - \overline{P}\overline{a}_{x+40:y+40}, \qquad (7.15)
$$

$$
{}_{60}V = \Big(\overline{a}_{\overline{x+60:y+60}} - \overline{a}_{x+60:y+60}\Big) - \overline{P}\overline{a}_{x+60:y+60}. \qquad (7.16)
$$

We can seperate the reserves at time t into two parts, benefits and premium incomes. Compared to the reserve at time 10, the reserve at time 40 is bigger because the income part of (7.15) is smaller than that of (7.14). However, the reserve at time 60 is small compared with the reserve of time 40. In spite of the similar income amount, the benefit part of (7.16) is much smaller than that of (7.15).

The expected present value of the annuity relating the premium payment period, $\overline{a}_{x+t:y+t}$, decrease as t gets larger. This makes the reserve decrease, since it has a negative coefficient in the reserves calculation. For this reason, the reserves escalates for a while as time passes. However, after a certain point, t is around 35 in our case, reserve decreases although the value of annuity relating premium payment period

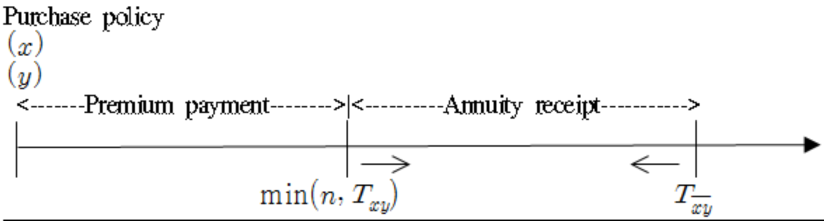


Figure 7.7: Cash flow and premium formula for Product 4

goes to zero. This is because the value of the annuity relating benefit annuity also decrease. In other words, after a certain amount of time passes, the insurer would not need to hold much reserve when the insureds get old, because both the premium payment period and the benefit annuity period become short. The reserves for other cases are skipped, as it is easily infered from section 6.2.4, which contains the reserve of the annuity given condition that one of the insureds has died prior to t.

7.4 Product analysis 4

In this subsection, we will analyze an adjusted version of Product 3 which is more realistic. Let us suppose that there is a couple who wants to recieve an annuity after their retirement point n, or the time of death of spouse occur. To meet their needs, insurance companies can modify the terms of Product 3 which especially related to premium payment period. This modified Product is called Product 4 and its cashflow is depicted in Figure 7.7. The insureds pay premiums continuously until the first death or n years, whichever comes first. This can be written as $min\,(T_{xy}, n)$. After the premium payment, an annuity is continuously paid by insurer until the second death. Many retirement annuity plans have similar cash flow structures. Let us assume that a couple both aged 40 purchase the above insurance with $n = 20$ years.

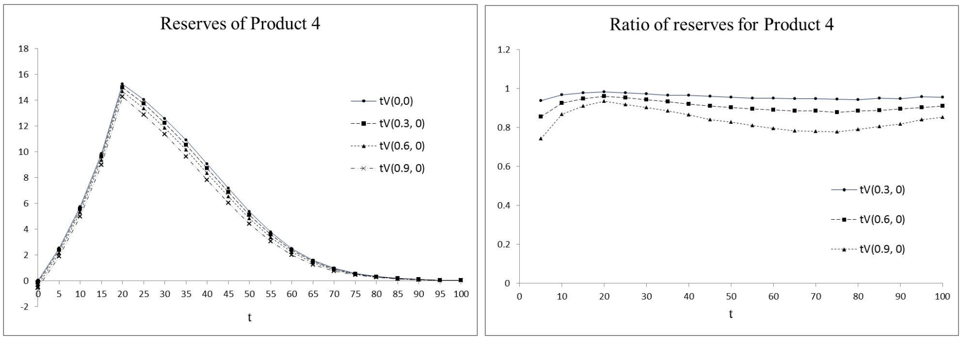


Figure 7.8: Reserves of Product 4 at time t (Assumption : $T_{40:40} > t$)

Table 7.4: Reserves of Product 4 at time t (Assumption : $T_{40:40} > t$)

	Time	0	0.1	5	10	15	20	40	60
Value	$tV(0.0, 0)$	0	0.056	2.548	5.763	9.869	15.277	9.059	2.519
	$tV(0.3, 0)$	-0.123	-0.08	2.389	5.574	9.649	15.011	8.746	2.393
	$tV(0.6, 0)$	-0.286	-0.241	2.178	5.329	9.366	14.693	8.339	2.241
	$tV(0.9, 0)$	-0.542	-0.474	1.895	4.997	8.993	14.279	7.837	2.004
Ratio	$tV(0.3, 0)$		-1.434	0.938	0.967	0.978	0.983	0.965	0.950
	$tV(0.6, 0)$		-4.318	0.855	0.925	0.949	0.962	0.920	0.890
	$tV(0.9, 0)$		-8.515	0.744	0.867	0.911	0.935	0.865	0.795

Using the equivalence principle of premium, the insurer can determine a premium at issue by

$$\overline{P} = \frac{\overline{a}_{\overline{xy}} - \overline{a}_{xy:\overline{n}|}}{\overline{a}_{xy:\overline{n}|}}. \tag{7.17}$$

As before, the loss random variable can be divided into three cases, but we focus on the case that both insureds are alive, in which case,

$${}_tL = \left(\overline{a}_{\overline{T_{\overline{xy}}-t}|} - \overline{a}_{\overline{min(T_{xy}-t,n-t)}|}\right) - \overline{P}\,\overline{a}_{\overline{min(T_{xy}-t,n-t)}|} \,|T_{xy} > t, \tag{7.18}$$

where $t < n$, and

$$ {}_tL = \left(\bar{a}_{\overline{T_{\overline{xy}}-t}|} - \bar{a}_{\overline{T_{xy}-t}|}\right) |T_{xy} > t, \tag{7.19} $$

where $t > n$. The other cases can be easily inferred from the previous section.

Figure 7.8 and table 7.4 are calculated under the assumption that both insureds are alive. In comparison with Product 3, Product 4 has adjusted premium payment period. The reserves of Product 4 can be analyzed by two periods: One before the premium payment has not yet finished, and the other period after the annuity initiates. Under the Product policy, after 20 years from the issue, the insurer must pay the annuity to the insureds regardless of the joint life status, unless both are dead by then. Therefore, the reserves steeply grow for the first 20 years for paying the benefit annuity. Once the 20^{th} year passes, the reserves of this Product is similar to the reserves of the last survivor annuity in section 6.2.3.

Appendix A

Experience Life Table in Korea

Man		Woman		Man		Woman	
Age	l_x	Age	l_x	Age	l_x	Age	l_x
0	100000.000	0	100000.000	25	98767.642	25	98604.965
1	99633.000	1	99487.000	26	98719.246	26	98570.453
2	99448.679	2	99233.308	27	98668.899	27	98534.968
3	99332.324	3	99074.535	28	98616.604	28	98499.495
4	99268.751	4	98991.312	29	98563.351	29	98462.065
5	99240.956	5	98957.655	30	98510.127	30	98423.665
6	99231.032	6	98948.749	31	98456.932	31	98383.311
7	99225.078	7	98946.770	32	98401.796	32	98341.007
8	99217.140	8	98940.833	33	98344.723	33	98295.770
9	99206.226	9	98930.939	34	98282.766	34	98248.588
10	99195.314	10	98921.046	35	98214.950	35	98198.481
11	99184.402	11	98911.154	36	98140.307	36	98146.436
12	99172.500	12	98901.263	37	98058.851	37	98091.474
13	99159.607	13	98891.373	38	97968.636	38	98033.600
14	99143.742	14	98879.506	39	97870.668	39	97973.799
15	99124.905	15	98866.651	40	97764.968	40	97911.096
16	99102.106	16	98851.821	41	97650.582	41	97846.475
17	99075.348	17	98834.028	42	97527.543	42	97778.961
18	99045.626	18	98813.273	43	97394.905	43	97707.582
19	99012.941	19	98790.546	44	97251.735	44	97632.347
20	98977.296	20	98764.860	45	97098.077	45	97551.312
21	98939.685	21	98736.219	46	96932.039	46	97465.467
22	98900.109	22	98705.610	47	96751.746	47	97372.875
23	98858.571	23	98673.037	48	96553.405	48	97274.528
24	98814.084	24	98639.489	49	96335.194	49	97168.499

Man		Woman		Man		Woman	
Age	l_x	Age	l_x	Age	l_x	Age	l_x
50	96093.393	50	97055.784	81	56339.613	81	75939.354
51	95826.253	51	96935.435	82	52659.510	82	73134.913
52	95530.150	52	96806.510	83	48742.169	83	69987.918
53	95204.392	53	96670.013	84	44647.339	84	66522.116
54	94846.424	54	96525.975	85	40435.756	85	62761.621
55	94455.656	55	96373.464	86	36160.887	86	58720.400
56	94028.717	56	96211.556	87	31875.822	87	54410.323
57	93563.275	57	96038.376	88	27643.988	88	49859.988
58	93056.162	58	95852.061	89	23539.132	89	45119.799
59	92504.339	59	95648.855	90	19635.873	90	40252.275
60	91905.836	60	95425.993	91	16003.826	91	35326.604
61	91255.142	61	95180.748	92	12704.477	92	30420.445
62	90546.090	62	94908.531	93	9790.832	93	25631.354
63	89772.826	63	94606.722	94	7299.653	94	21069.486
64	88928.064	64	94270.868	95	5243.998	95	16840.629
65	88005.880	65	93897.556	96	3612.485	96	13035.489
66	86998.213	66	93480.650	97	2372.897	97	9724.866
67	85894.205	67	93011.378	98	1476.417	98	6955.030
68	84682.238	68	92480.283	99	863.571	99	4740.061
69	83352.727	69	91877.311	100	470.776	100	3057.671
70	81896.555	70	91193.744	101	236.998	101	1852.582
71	80307.762	71	90422.245	102	109.156	102	1045.412
72	78585.963	72	89558.712	103	45.551	103	544.241
73	76741.551	73	88606.703	104	17.031	104	258.466
74	74788.478	74	87574.435	105	5.625	105	110.455
75	72732.543	75	86461.364	106	1.611	106	41.790
76	70554.931	76	85246.582	107	0.390	107	13.735
77	68213.212	77	83886.899	108	0.077	108	3.832
78	65657.263	78	82326.603	109	0.013	109	0.885
79	62841.880	79	80510.478	110	0.002	110	0.164
80	59738.748	80	78392.247	111	0.000	111	0.024
						112	0.003
						113	0.000

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